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2010 J. Phys. A: Math. Theor. 43 265301

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Brownian motion on Lie groups and open quantum systems

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Received 4 March 2010

Published 2 June 2010

Online at stacks.iop.org/JPhysA/43/265301

Abstract

We study the *twirling semigroups* of (super) operators, namely certain quantum dynamical semigroups that are associated, in a natural way, with the pairs formed by a projective representation of a locally compact group and a convolution semigroup of probability measures on this group. The link connecting this class of semigroups of operators with (classical) Brownian motion is clarified. It turns out that every twirling semigroup associated with a finite-dimensional representation is a random unitary semigroup, and, conversely, every random unitary semigroup arises as a twirling semigroup. Using standard tools of the theory of convolution semigroups of measures and of convex analysis, we provide a complete characterization of the infinitesimal generator of a twirling semigroup associated with a finite-dimensional unitary representation of a Lie group.

PACS numbers: 03.65.Yz, 03.65.Ca, 03.65.Aa, 03.67.—a

1. Introduction

The theory of Brownian motion and its several ramifications form an evergreen area of research in physics and mathematics. The interesting history of this subject would deserve a whole article *per se*; hence, we will content ourselves with recalling just a few salient facts related to our present contribution. The first investigations of Brownian motion on a Lie group—and, more generally, of probability theory on groups—seem to be due to Perrin [1], who studied Brownian motion on the rotation group $SO(3)$, and, later, to Lévy [2] who provided the first theoretical treatment of probability measures on $U(1)$ (also consider the early work of

von Mises [3] who, studying the atomic weights, introduced a normal distribution on the torus). These investigations paved the way to an extensive study of probability theory on locally compact groups (started in the 1940s); see the classical references [4, 5], and the rich bibliography therein. In particular, fundamental and systematic contributions to the theory of Brownian motion on Lie groups are due to Ito [6], Yosida [7] and Hunt [8].

In 1966, Nelson showed that there is a remarkable link connecting (classical) Brownian motion and the Schrödinger equation [9]. Assuming that a particle of mass m is subject to a Brownian motion with diffusion coefficient $\hbar/2m$ (and no friction), and using the well-known relation between the particle probability density and the quantum–mechanical wavefunction, he was able to derive (formally) the Schrödinger equation.

A different association of the evolution of a quantum system with Brownian motion was proposed, later on, by Kossakowski [10]. In the pioneering times of the theory of *open* quantum systems [11]—a complete definition of quantum dynamical semigroups and the Gorini–Kossakowski–Lindblad–Sudarshan classification of the infinitesimal generators [12, 13] has not been established yet—he observed that there is a class of semigroups of (super) operators—acting in a space of trace-class operators—that are generated, in a natural way, by the pairs of the type $(U, \{\mu_t\}_{t \in \mathbb{R}^+})$, where U is a representation of a group G and $\{\mu_t\}_{t \in \mathbb{R}^+}$ is a convolution semigroup of measures on G . In particular, he considered the case where G is a Lie group and $\{\mu_t\}_{t \in \mathbb{R}^+}$ is what we call nowadays a *Gaussian* semigroup of measures (see section 5). This class of convolution semigroups of measures describe the statistical properties of Brownian motion on G (the natural generalization of the ordinary Brownian motion).

The aim of this paper is to provide a rigorous study of the above-mentioned class of semigroups of superoperators—that we will call *twirling semigroups*—without restrictions on the convolutions semigroups of measures considered. In particular, in the case where G is a Lie group, we will not assume, in general, to deal with Gaussian semigroups of measures. We will prove that every twirling semigroup is a *quantum dynamical semigroup* [14], and, in the case where G is a Lie group and U is a finite-dimensional unitary representation, we will provide a complete characterization of the infinitesimal generators of the twirling semigroups associated with U .

Like many other mathematical objects having a ‘natural’ definition, it turns out that twirling semigroups arise in the study of various physical contexts. For instance, the analysis of the infinitesimal generators of the twirling semigroups reveals that this class of semigroups of superoperators includes, in particular, the semigroups describing the dynamics of a finite-dimensional system with a purely random Gaussian stochastic Hamiltonian [15], and the reduced dynamics of a finite-dimensional system in the limit of singular coupling to a reservoir at infinite temperature [16].

The twirling semigroups associated with the defining representation of the group $SU(N)$ have been studied by Kümmerer and Maassen [17], with the aim of characterizing the dilations of dynamical semigroups that are ‘essentially commutative’.

Our interest in twirling semigroups is also motivated by possible applications in the field of quantum computation and information [18], where, usually, finite-dimensional quantum systems are considered. In fact, it is well known that a relevant class of ‘quantum channels’ is formed by the so-called *random unitary maps*, i.e. by those completely positive trace-preserving maps that can be expressed as convex superpositions of unitary transformations. Gregoratti and Werner [19] have given a remarkable characterization of this class of maps: they are the only quantum channels that enjoy the property of being *perfectly corrigible* by using, as the only side resource, classical information obtained from the environment. Smolin, Verstraete and Winter [20] have conjectured that asymptotically many copies of any unital quantum channel (a quantum bistochastic map [21])—random unitary maps form a subset of

the set of unital channels—may be arbitrarily well approximated by a random unitary map. This conjecture, if proved, would be a ‘quantum counterpart’ of the Birkhoff–von Neumann theorem [22] on bistochastic matrices. Recently, Mendl and Wolf [23] have studied the relation between the set of unital channels and the subset of random unitary maps, and verified the conjecture in special cases. Other recent investigations of random unitary maps include applications to quantum cryptography [24] and quantum state reconstruction [25].

It is therefore an interesting and natural issue to characterize the *random unitary semigroups*, i.e. the quantum dynamical semigroups consisting of random unitary maps. But it turns out that—in the case of a finite-dimensional quantum system—there is a precise relation between random unitary semigroups and twirling semigroups: indeed, every twirling semigroup is a random unitary semigroup—see section 5—and, conversely, it can be shown that every random unitary semigroup arises as a twirling semigroup. Thus, it is likely that our results—in addition to their intrinsic theoretical interest—may find useful applications in the context of quantum information.

The paper is organized as follows. In section 2, for the reader’s convenience, we will recall some mathematical facts that are fundamental in the rest of this paper, and we will set the main definitions and notations. Some further notations will be introduced later on, closer to the place where they are used. In section 3, we will briefly discuss the group-theoretical framework underlying the description of the statistical properties of ‘standard’ Brownian motion. This should help the reader to achieve a clearer understanding of the general framework. In section 4, the main object of our investigation—the twirling semigroups—will be introduced, where the basic properties of these semigroups of superoperators will be studied. In section 5, we will focus on the case of twirling semigroups associated with finite-dimensional representations of Lie groups. As already mentioned, this case is relevant for applications to quantum information. Eventually, in section 6, a few conclusions will be drawn.

2. Definitions, basic known facts and notations

In this section, we fix the main notations, and recall some basic definitions and results that will be useful in the rest of this paper. We will be rather concise and, for further details, we invite the reader to consult the standard references [26] (functional analysis and basics in probability theory), [27, 28] (semigroups of operators), [29, 30] (Lie groups, representation theory) and [4, 5] (probability theory on groups).

Let \mathfrak{X} be a separable real or complex Banach space. Denoting by \mathbb{R}^+ the set of non-negative real numbers (the set of strictly positive real numbers will be denoted by \mathbb{R}_*^+), a family $\{\mathfrak{C}_t\}_{t \in \mathbb{R}^+}$ of bounded linear operators in \mathfrak{X} is said to be a (one-parameter) *semigroup of operators* if the following conditions are satisfied:

- (i) $\mathfrak{C}_t \mathfrak{C}_s = \mathfrak{C}_{t+s}, \forall t, s \geq 0$ (one-parameter semigroup property);
- (ii) $\mathfrak{C}_0 = I$;
- (iii) $\lim_{t \downarrow 0} \|\mathfrak{C}_t \zeta - \zeta\| = 0, \forall \zeta \in \mathfrak{X}$, i.e. $s\text{-}\lim_{t \downarrow 0} \mathfrak{C}_t = I$ (strong right continuity at $t = 0$).

Here and throughout the paper, I is the identity operator. According to a classical result—see [27]—the previous conditions imply that the map $\mathbb{R}^+ \ni t \mapsto \mathfrak{C}_t \in \mathfrak{X}$ is strongly continuous. Moreover [28], the last condition is equivalent to the assumption that $w\text{-}\lim_{t \downarrow 0} \mathfrak{C}_t = I$ (weak limit). A semigroup of operators $\{\mathfrak{C}_t\}_{t \in \mathbb{R}^+}$ is said to be a *contraction semigroup* if, in addition to the previous hypotheses, it satisfies

- (iv) $\|\mathfrak{C}_t\| \leq 1, \forall t > 0$.

A semigroup of operators $\{\mathcal{C}_t\}_{t \in \mathbb{R}^+}$ admits a densely defined *infinitesimal generator*, namely the closed linear operator \mathfrak{A} in \mathfrak{X} defined by

$$\begin{aligned} \text{Dom}(\mathfrak{A}) &:= \left\{ \zeta \in \mathfrak{X} : \exists \lim_{t \downarrow 0} t^{-1}(\mathcal{C}_t \zeta - \zeta) \right\}, \\ \mathfrak{A}\zeta &:= \lim_{t \downarrow 0} t^{-1}(\mathcal{C}_t \zeta - \zeta), \quad \forall \zeta \in \text{Dom}(\mathfrak{A}). \end{aligned} \tag{2.1}$$

Let X be a locally compact, second countable, Hausdorff topological space (in short, l.c.s.c. space). If X is noncompact, the symbol \dot{X} will indicate the one-point compactification of X . We will denote by $C_0(X)$ the Banach space of all continuous \mathbb{R} -valued functions on X vanishing at infinity (hence, bounded), endowed with the ‘sup-norm’:

$$\|f\|_{\text{sup}} := \sup_{x \in X} |f(x)|, \quad f \in C_0(X). \tag{2.2}$$

As is well known, $C_0(X)$ is the closure, with respect to the sup-norm, of the vector space $C_c(X)$ of all continuous \mathbb{R} -valued functions on X with compact support. If X is noncompact, the vector space $C_0(X)$ can be immersed in a natural way in $C(\dot{X})$, the Banach space of continuous real-valued functions on \dot{X} (endowed with the sup-norm) — i.e. setting $f(\infty) = 0$, for all $f \in C_0(X)$ —and every function in $C(\dot{X})$ can be expressed as the sum of a function in $C_0(X)$ and a constant function.

We will call a contraction semigroup $\{\mathcal{C}_t\}_{t \in \mathbb{R}^+}$ in the Banach space $C_0(X)$ a *Markovian semigroup* if it satisfies the conditions

$$C_0(X) \ni f \geq 0 \implies \mathcal{C}_t f \geq 0, \quad \forall t > 0, \tag{2.3}$$

(hence, $\mathcal{C}_t f_1 \geq \mathcal{C}_t f_2$ for $f_1 \geq f_2$)—thus, for each $x \in X$, the map $F_{t;x} : C_0(X) \ni f \mapsto (\mathcal{C}_t f)(x)$ must be a (bounded) positive functional, with $\|F_{t;x}\| \leq \|\mathcal{C}_t\| \leq 1$ —and

$$\sup_{f \in C_0(X), 0 \leq f \leq 1} (\mathcal{C}_t f)(x) = 1, \quad \forall x \in X, \quad \forall t > 0, \tag{2.4}$$

i.e. $\|F_{t;x}\| \geq 1$; hence $\|F_{t;x}\| = 1$. Clearly, condition (2.4) implies that the contraction semigroup $\{\mathcal{C}_t\}_{t \in \mathbb{R}^+}$ is such that $\|\mathcal{C}_t\| = 1$, for all $t \geq 0$. Moreover, by the Riesz representation theorem there exists a unique family $\{\mathfrak{p}_{t;x} : t \in \mathbb{R}^+, x \in X\}$ of (regular)⁴ probability measures on X such that

$$(\mathcal{C}_t f)(x) = F_{t;x} f = \int_X f(y) d\mathfrak{p}_{t;x}(y), \quad \forall f \in C_0(X), \quad \forall x \in X, \quad \forall t \geq 0. \tag{2.5}$$

Assume, in particular, that the topological space X is compact. Then, $1 \in C_0(X)$ ($= C(X)$), and $1 = \|F_{t;x}\| = F_{t;x} 1$, for all $x \in X$ and $t \geq 0$. Therefore, in this case, condition (2.4) can be replaced by the following:

$$\mathcal{C}_t 1 = 1, \quad \forall t > 0. \tag{2.6}$$

We will denote by $C_c^2(\mathbb{R}^n)$ the vector space of all \mathbb{R} -valued functions on \mathbb{R}^n , ‘of class C^2 ’, with compact support. The completion of this vector space, with respect to the norm

$$\|f\|_{\diamond} := \|f\|_{\text{sup}} + \sum_{j=1}^n \left\| \frac{\partial}{\partial x^j} f \right\|_{\text{sup}} + \sum_{j,k=1}^n \left\| \frac{\partial^2}{\partial x^j \partial x^k} f \right\|_{\text{sup}}, \tag{2.7}$$

is a real Banach space $C_0^2(\mathbb{R}^n)$ (it is clear that $C_0^2(\mathbb{R}^n) \subset C_0(\mathbb{R}^n)$). Moreover, we will denote by $C^2(\mathbb{R}^n)$ the completion with respect to the norm $\|\cdot\|_{\diamond}$ of the real vector space consisting of all linear superpositions of functions in the vector space $C_c^2(\mathbb{R}^n)$ and constant functions on

⁴ Recall that in a l.c.s.c. space every finite Borel measure is regular.

\mathbb{R}^n ; i.e. $C^2(\mathbb{R}^n)$ is the direct sum of $C_0^2(\mathbb{R}^n)$ and the one-dimensional vector space of constant real-valued functions on \mathbb{R}^n .

Complexifications of some of the real vector spaces of functions introduced above will also be considered. For instance, we will consider the complexification $C_c(X; \mathbb{C})$ of the real vector space $C_c(X) \equiv C_c(X; \mathbb{R})$. The notations adopted will be consistent with this example.

Let G be a locally compact, second countable, Hausdorff topological group (in short, l.c.s.c. group). The symbol e will denote the identity in G , and G_* the set $G \setminus \{e\}$.

By the term *projective representation* of G we mean a Borel projective representation of G in a separable complex Hilbert space \mathcal{H} (see, for instance, [30], chapter VII), namely a map U of G into $\mathcal{U}(\mathcal{H})$ —the unitary group of \mathcal{H} (the group of all unitary operators in \mathcal{H})—such that

- U is a weakly Borel map, i.e. $G \ni g \mapsto \langle \phi, U(g)\psi \rangle \in \mathbb{C}$ is a Borel function, for any pair of vectors $\phi, \psi \in \mathcal{H}$;
- $U(e) = I$;
- denoting by \mathbb{T} the circle group, namely the group of complex numbers of modulus one, there exists a Borel function $m : G \times G \rightarrow \mathbb{T}$ such that

$$U(gh) = m(g, h) U(g) U(h), \quad \forall g, h \in G. \tag{2.8}$$

The function m is called the *multiplier associated with U* (multipliers, however, will play no relevant role in our later discussion). Clearly, in the case where $m \equiv 1$, U is a standard unitary representation; in this case, according to a well-known result, the hypothesis that the map U is weakly Borel implies that it is, actually, strongly continuous.

We will denote by $\mathcal{M}^1(G)$ the semigroup—with respect to convolution of measures⁵—of all (regular) probability measures on G , endowed with the weak topology (which, in $\mathcal{M}^1(G)$, coincides with the vague topology). The symbol $\delta \equiv \delta_e$ will denote the Dirac measure at e , measure that is, of course, the identity in the semigroup $\mathcal{M}^1(G)$. By a *continuous convolution semigroup of measures* on G we mean a subset $\{\mu_t\}_{t \in \mathbb{R}^+}$ of $\mathcal{M}^1(G)$ such that the map $\mathbb{R}^+ \ni t \mapsto \mu_t \in \mathcal{M}^1(G)$ is a homomorphism of semigroups and

$$\lim_{t \downarrow 0} \mu_t = \delta. \tag{2.9}$$

It is a well-known fact that this condition implies that the homomorphism $t \mapsto \mu_t$ is continuous. Let μ be a probability measure in $\mathcal{M}^1(G)$. The *probability operator* associated with μ is a bounded linear operator $\mathfrak{P}_\mu : C_0(G) \rightarrow C_0(G)$ defined by

$$(\mathfrak{P}_\mu f)(g) := \int_G f(gh) d\mu(h) = \int_G f(h) d\mu_g(h), \quad \forall f \in C_0(G), \tag{2.10}$$

where the probability measure μ_g is the g -translate of the measure μ . The probability operator \mathfrak{P}_μ is *left invariant*, i.e.

$$\mathfrak{P}_\mu \ell_g = \ell_g \mathfrak{P}_\mu, \quad \forall g \in G, \tag{2.11}$$

where $\ell_g : C_0(G) \rightarrow C_0(G)$ is the isometry defined by $\ell_g f := f(g \cdot)$, and $\|\mathfrak{P}_\mu\| = 1$ (by one of the assertions of the Riesz representation theorem). Note, moreover, that the correspondence between probability measures and probability operators is one-to-one.

A convolution semigroup of measures on G generates, in a natural way, a contraction semigroup. Precisely, let $\{\mu_t\}_{t \in \mathbb{R}^+}$ be a continuous convolution semigroup of measures on G . Then, setting

$$\mathfrak{P}_t := \mathfrak{P}_{\mu_t}, \quad t \geq 0, \quad (\mathfrak{P}_0 = I), \tag{2.12}$$

⁵ Recall that for $\mu, \nu \in \mathcal{M}^1(G)$ the convolution of μ with ν is the measure $\mu \star \nu \in \mathcal{M}^1(G)$ determined by $\int_G d\mu \star \nu(g) f(g) = \int_G d\mu(g) \int_G d\nu(h) f(gh)$ for all $f \in C_c(G)$.

we get a contraction semigroup $\{\mathfrak{P}_t\}_{t \in \mathbb{R}^+}$ —precisely, a Markovian semigroup—in the Banach space $C_0(G)$, which is left invariant: $\mathfrak{P}_t \ell_g = \ell_g \mathfrak{P}_t$, for all $g \in G$ and $t \in \mathbb{R}^+$. A semigroup of operators of the type (2.12) will be called a *probability semigroup* on G . Actually, it turns out that definition (2.12) establishes a one-to-one correspondence between the left-invariant Markovian semigroups in $C_0(G)$ and the continuous convolution semigroups of measures on G (or the associated probability semigroups).

Now let G be, in particular, a Lie group of dimension $n \geq 1$. We will denote by $BC^\infty(G)$, $C_c^\infty(G)$ the vector spaces of all bounded smooth real-valued functions on G and of all smooth real-valued functions on G with compact support, respectively. For every basis $\{\xi_1, \dots, \xi_n\}$ in the Lie algebra $\text{Lie}(G)$ (realized as the space of left-invariant vector fields) of G , there exists a relatively compact neighborhood \mathcal{E}_e of the identity in G and a local chart

$$(\mathcal{E}_e; \mathcal{E}_e \ni g \mapsto x^1(g), \dots, \mathcal{E}_e \ni g \mapsto x^n(g)), \tag{2.13}$$

such that $\exp_G\left(\sum_{k=1}^n x^k(g)\xi_k\right) = g$, for all $g \in \mathcal{E}_e$. Such a local chart is called a *system of canonical coordinates* (of the first kind) associated with the basis $\{\xi_1, \dots, \xi_n\}$. The local maps $g \mapsto x^1(g), \dots, g \mapsto x^n(g)$ defined in \mathcal{E}_e can be extended to suitable real functions

$$G \ni g \mapsto \bar{x}^1(g) \in \mathbb{R}, \dots, G \ni g \mapsto \bar{x}^n(g) \in \mathbb{R}, \tag{2.14}$$

belonging to $C_c^\infty(G)$. We will call such a set of real functions a *system of adapted coordinates* (based at the identity e) for the Lie group G .

Let U be a *smooth unitary* representation⁶ of the Lie group G in a *finite-dimensional* Hilbert space \mathcal{H} . Then, there is a unique representation π_U of the Lie algebra $\text{Lie}(G)$ in \mathcal{H} determined by

$$U(\exp_G(\xi)) = e^{\pi_U(\xi)}, \quad \forall \xi \in \text{Lie}(G). \tag{2.15}$$

It is clear that $\pi_U(\text{Lie}(G)) \subset i\mathcal{B}_{\mathbb{R}}(\mathcal{H})$, with $i\mathcal{B}_{\mathbb{R}}(\mathcal{H})$ denoting the finite-dimensional real vector space consisting of all skew-adjoint operators in \mathcal{H} (accordingly, the real vector space of self-adjoint operators in \mathcal{H} will be denoted by $\mathcal{B}_{\mathbb{R}}(\mathcal{H})$). We will adopt the following notation:

$$\hat{X}_1 \equiv \pi_U(\xi_1), \dots, \hat{X}_n \equiv \pi_U(\xi_n). \tag{2.16}$$

Observe that the map $G \ni g \mapsto e^{\pi_U(\bar{x}^1(g)\xi_1 + \dots + \bar{x}^n(g)\xi_n)} \in \mathcal{B}(\mathcal{H})$ is a smooth function such that

$$U(g) = e^{\pi_U(\bar{x}^1(g)\xi_1 + \dots + \bar{x}^n(g)\xi_n)} = e^{\bar{x}^1(g)\hat{X}_1 + \dots + \bar{x}^n(g)\hat{X}_n}, \quad \forall g \in \mathcal{E}_e. \tag{2.17}$$

We will now recall a classical result about left-invariant Markovian semigroups (probability semigroups) [4, 5]. Let $\Xi \equiv \{\xi_1, \dots, \xi_n\}$ be a basis in $\text{Lie}(G)$. A *Hunt function* associated with Ξ is a real-valued function on G that verifies the following conditions: it is a function Φ contained in $BC^\infty(G)$, with $0 < \Phi \leq 1$, such that

$$\Phi(g) = \sum_{j=1}^n x^j(g)^2, \quad \forall g \in \mathcal{E}_e \quad \text{and} \quad \Phi(g) = 1, \quad \forall g \in \mathcal{K}_e, \tag{2.18}$$

where \mathcal{E}_e is a relatively compact neighborhood of e , $(\mathcal{E}_e; \mathcal{E}_e \ni g \mapsto x^1(g), \dots, \mathcal{E}_e \ni g \mapsto x^n(g))$ a system of canonical coordinates (extendable to adapted coordinates denoted as in (2.14)) associated with the basis Ξ and \mathcal{K}_e a compact neighborhood of the identity. A *Lévy measure* η is a Radon measure on G_* satisfying

$$\int_{G_*} \Phi(g) d\eta(g) < \infty, \tag{2.19}$$

⁶ As is well known, a continuous homomorphism between Lie groups is necessarily smooth. Therefore, it would be enough to assume continuity in order to ensure smoothness.

for any Hunt function Φ . Let us denote by \mathfrak{J} the infinitesimal generator of a probability semigroup $\{\mathfrak{P}_t\}_{t \in \mathbb{R}^+}$ in $C_0(G)$. Then, the domain of the operator \mathfrak{J} contains the vector space $C_c^2(G)$, and there exist real numbers b^1, \dots, b^n , a positive⁷, symmetric real matrix $[a^{jk}]_{j,k=1}^n$ and a Lévy measure η on G_* such that

$$(\mathfrak{J}f)(g) = \sum_{j=1}^n b^j (\xi_j f)(g) + \sum_{j,k=1}^n a^{jk} (\xi_j \xi_k f)(g) + (\mathfrak{R}f)(g) \tag{2.20}$$

for all $f \in C_c^2(G)$, where

$$(\mathfrak{R}f)(g) = \int_{G_*} \left(f(gh) - f(g) - \sum_{j=1}^n (\xi_j f)(g) \bar{x}^j(h) \right) d\eta(h). \tag{2.21}$$

This result is the celebrated *Lévy–Kintchine formula*. If $\{\mu_t\}_{t \in \mathbb{R}^+}$ is the continuous convolution semigroup of measures that generates the probability semigroup $\{\mathfrak{P}_t\}_{t \in \mathbb{R}^+}$, then the Lévy measure η is uniquely determined by the condition

$$\int_{G_*} f(g) d\eta(g) = \lim_{t \downarrow 0} t^{-1} \int_G f(g) d\mu_t(g), \quad \forall f \in C_c(G_*), \quad (f(e) \equiv 0). \tag{2.22}$$

Conversely—given real numbers b^1, \dots, b^n , a positive, symmetric real matrix $[a^{jk}]_{j,k=1}^n$ and a Lévy measure η on G_* —one can prove that there is a probability semigroup $\{\mathfrak{P}_t\}_{t \in \mathbb{R}^+}$ whose infinitesimal generator satisfies the Lévy–Kintchine formula (2.20). Therefore, it is natural to call a set $\{b^j, a^{jk}, \eta\}_{j,k=1}^n \equiv \{b^1, \dots, b^n; [a^{jk}]_{j,k=1}^n; \eta\}$ of the type just described as a *representation kit* (this term is non-standard) of the probability semigroup $\{\mathfrak{P}_t\}_{t \in \mathbb{R}^+}$; or—due to the one-to-one correspondence between the continuous convolution semigroups of measures and probability semigroups—a representation kit of the convolution semigroup $\{\mu_t\}_{t \in \mathbb{R}^+}$ (generating $\{\mathfrak{P}_t\}_{t \in \mathbb{R}^+}$).

Remark 2.1. Since the Lévy–Kintchine formula (2.20) has been written for functions in $C_c^2(G)$ —that is perfectly fit for our purposes—we can use the standard Lie derivatives ξ_1, \dots, ξ_n of functions on G instead of the ‘uniform derivatives’ (i.e. derivatives converging in the sup-norm, defined on suitable Banach spaces), as it is usually done in more general contexts [4, 5].

A probability semigroup $\{\mathfrak{P}_t\}_{t \in \mathbb{R}^+}$ acting in $C_0(G) \equiv C_0(G; \mathbb{R})$ can be extended, in a natural way, to $C_0(G; \mathbb{C})$ ‘by complexification’ and the infinitesimal generator of this extended semigroup is the complexification of the generator \mathfrak{J} of $\{\mathfrak{P}_t\}_{t \in \mathbb{R}^+}$. With a slight abuse, we will still denote by \mathfrak{J} the complexified generator, and the Lévy–Kintchine formula (2.20) will be understood to hold, in general, in $C_c(G; \mathbb{C})$.

It is convenient to classify convolution semigroups of measures on Lie groups according to the behavior of the associated Lévy measures. We will say that $\{\mu_t\}_{t \in \mathbb{R}^+}$ is of *regular type* if the associated Lévy measure η satisfies

$$\int_{G_*} \sum_{j=1}^n |\bar{x}^j(g)| d\eta(g) < \infty. \tag{2.23}$$

This condition does not depend on the choice of the adapted coordinates. Note that, if (2.23) is verified, we have

$$(\mathfrak{R}f)(g) = \int_{G_*} (f(gh) - f(g)) d\eta(h) - \sum_{j=1}^n (\xi_j f)(g) \int_{G_*} \bar{x}^j(h) d\eta(h). \tag{2.24}$$

⁷ In the following, by *positive* we will always mean positive semidefinite.

We will, moreover, single out a special class of convolution semigroups of the measures of regular type. We will say that the convolution semigroup of measures $\{\mu_t\}_{t \in \mathbb{R}^+}$ is of the *first kind* if the associated Lévy measure η on G_* is finite (hence, satisfies (2.23)). Otherwise, we will say that it is a convolution semigroup of measures of *second kind*. Clearly, the convolution semigroups of measures of the second kind that are of nonregular type are characterized by Lévy measures satisfying (2.19) but not the more stringent condition (2.23).

Let \mathcal{A} be a C^* -algebra. We recall that a bounded linear map $\Phi : \mathcal{A} \rightarrow \mathcal{A}$ is said to be *completely positive* if the map $\Phi \otimes I_M : \mathcal{H} \otimes \mathbb{C}^M \rightarrow \mathcal{H} \otimes \mathbb{C}^M$ —with I_M denoting the identity operator in \mathbb{C}^M —is positive for any $M \in \mathbb{N}$. As is well known, in the case where $\mathcal{A} = \mathcal{B}(\mathcal{H})$ —the C^* -algebra of all bounded linear maps in a separable complex Hilbert space \mathcal{H} —and $\dim(\mathcal{H}) = N < \infty$, Φ is completely positive if and only if it is N -positive, i.e. $\Phi \otimes I_N$ is positive. It is also known (see, e.g., [31]) that the map Φ is N -positive if and only if, for every N -tuple $\{\psi_1, \dots, \psi_N\}$ in \mathcal{H} and every N -tuple $\{\hat{A}_1, \dots, \hat{A}_N\}$ in $\mathcal{B}(\mathcal{H})$,

$$\sum_{j,k=1}^N \langle \psi_j, \Phi(\hat{A}_j^* \hat{A}_k) \psi_k \rangle \geq 0. \tag{2.25}$$

3. The Brownian motion on \mathbb{R}^n

The aim of this section is to recall that the statistical properties of ‘standard’ Brownian motion—i.e. the Brownian motion on the Euclidean space \mathbb{R}^n —can be expressed, in a natural way, in the language of convolution semigroups of probability measures (technically, the distributions associated with the Wiener processes that are the mathematical formalization of Brownian motion [32]) and of the associated Markovian semigroups. In this case ($G = \mathbb{R}^n$), it will be instructive to consider a slightly more general mathematical context with respect to the one considered in section 2 for introducing the Lévy–Kintchine formula (2.20). This will help the reader, in particular, to appreciate the role of the invariance with respect to translations in our discussion. We will essentially follow the approach of Nelson’s classical book [33].

As is well known—see [34]—the evolution of the probability distribution of the position of a Brownian particle (in \mathbb{R}^n , $n \geq 1$), suspended in a viscous, infinitely extended fluid, can be regarded as the diffusion through the fluid of a unit mass initially concentrated at a point, let us say the origin of \mathbb{R}^n . If the relevant properties of the fluid are assumed to be invariant with respect to translations and the external forces acting on the Brownian particle are constant (with respect to space and time)—a constant force field that causes a constant (average) drift velocity of a particle in the fluid [35]—then by translating in \mathbb{R}^n any solution of the equations governing the diffusion process one must obtain another solution.

Let us mathematically formalize the diffusion process described above. We will start by considering the simplest case: a single degree of freedom and no drift. Let us consider, then, a family of probability measures $\{\mu_t\}_{t \in \mathbb{R}_*^+}$ on \mathbb{R} such that

$$\mu_t \star \mu_s = \mu_{t+s}, \quad t, s \in \mathbb{R}_*^+, \tag{3.1}$$

where we recall that $\mu_t \star \mu_s$ is the convolution of the measure μ_t with the measure μ_s . Suppose that for all $\epsilon > 0$,

$$\mu_t(\{y : |y| \geq \epsilon\}) = o(t), \quad t \downarrow 0. \tag{3.2}$$

Note that this assumption implies, in particular, that

$$\lim_{t \downarrow 0} \mu_t = \delta \quad (\text{weakly}). \tag{3.3}$$

Hence—setting $\mu_0 = \delta$ — $\{\mu_t\}_{t \in \mathbb{R}^+}$ is a continuous convolution semigroup of measures on \mathbb{R} . Suppose, moreover, that the measure μ_t is invariant with respect to the transformation $x \mapsto -x$ (\Leftrightarrow no drift). Then, it follows that either $\mu_t = \delta$, for all $t \in \mathbb{R}^+$ —there is no diffusion—or, for $t > 0$, μ_t is absolutely continuous with respect to the Lebesgue measure on \mathbb{R} and

$$d\mu_t(y) = \wp_t(y) dy = \frac{1}{\sqrt{4\pi Dt}} e^{-(y^2/4Dt)} dy, \quad t > 0, \tag{3.4}$$

for some $D > 0$ (diffusion constant). Thus, the Radon–Nikodým derivative \wp_t of the measure μ_t with respect to the Lebesgue measure satisfies the diffusion equation

$$\frac{\partial}{\partial t} \wp_t(y) = D \frac{\partial^2}{\partial y^2} \wp_t(y), \quad t > 0; \tag{3.5}$$

precisely, it is the fundamental solution of this equation. The translation-invariant semigroup (probability semigroup) $\{\mathfrak{P}_t\}_{t \in \mathbb{R}^+}$ associated with the semigroup of probability measures $\{\mu_t\}_{t \in \mathbb{R}^+}$ is given by

$$(\mathfrak{P}_t f)(x) := \int_{\mathbb{R}} f(x+y) d\mu_t(y) = \int_{\mathbb{R}} f(y) \wp_t(y-x) dy, \tag{3.6}$$

$$f \in C_0(\mathbb{R}), \quad t > 0, \quad (\mathfrak{P}_0 = I). \tag{3.6}$$

Clearly, for $f \geq 0$ and $t > 0$, $\mathfrak{P}_t f$ can be interpreted as the (expected) concentration, at time t , of a suspension of Brownian particles with initial ($t = 0$) concentration f . Note that one can extend, in a natural way, the domain of the operators in the semigroup $\{\mathfrak{P}_t\}_{t \in \mathbb{R}^+}$ to include linear superpositions with the constant functions in such a way to obtain a Markovian semigroup in the Banach space $C(\dot{\mathbb{R}}^n)$ ($\dot{\mathbb{R}}^n = \mathbb{R}^n \cup \infty$). Obviously, this Markovian semigroup commutes with translations.

Keeping in mind the ‘elementary case’ briefly sketched above, let us now consider a more general setting. We will focus on the implications of an assumption of type (3.2), without assuming, at first, invariance with respect to translations. Then, let $\{\mathfrak{C}_t\}_{t \in \mathbb{R}^+}$ be a Markovian semigroup in the Banach space $C(\dot{\mathbb{R}}^n)$, and let \mathfrak{A} be the associated infinitesimal generator. Suppose that

$$\text{Dom}(\mathfrak{A}) \supset C_c^2(\mathbb{R}^n) \tag{3.7}$$

(a technical condition), and, for all $x \in \mathbb{R}^n$ and all $\epsilon > 0$,

$$p_{t;x}(\{y \in \mathbb{R}^n : |y-x| \geq \epsilon\}) = o(t), \quad t \downarrow 0, \tag{3.8}$$

where $\{p_{t;x} : t \in \mathbb{R}^+, x \in \dot{\mathbb{R}}^n\}$ is the family of probability measures determined by (2.5), with $X = \dot{\mathbb{R}}^n$. Then, one can prove that there are continuous real-valued functions a^{jk} and b^j on \mathbb{R}^n , $j, k = 1, \dots, n$, such that

$$(\mathfrak{A}f)(x) = \sum_{j=1}^n b^j(x) \frac{\partial}{\partial x^j} f(x) + \sum_{j,k=1}^n a^{jk}(x) \frac{\partial^2}{\partial x^j \partial x^k} f(x), \quad \forall f \in C_c^2(\mathbb{R}^n), \quad \forall x \in \mathbb{R}^n. \tag{3.9}$$

Moreover, for each $x \in \mathbb{R}^n$, the matrix $[a^{jk}(x)]_{j,k=1}^n$ is positive, i.e.

$$\sum_{j,k=1}^n a^{jk}(x) z_j^* z_k \geq 0, \quad \forall z_1, \dots, z_n \in \mathbb{C}. \tag{3.10}$$

As the matrix $[a^{jk}(x)]_{j,k=1}^n$ may be singular, the operator \mathfrak{A} is not necessarily elliptic. It is clear that, in the case where the Markovian semigroup $\{\mathfrak{C}_t\}_{t \in \mathbb{R}^+}$ commutes with translations, i.e.

$$(\mathfrak{C}_t f)(x + (\cdot)) = \mathfrak{C}_t(f(x + (\cdot))), \quad f \in C(\dot{\mathbb{R}}^n), \quad x \in \mathbb{R}^n, \quad (x + \infty \equiv \infty), \tag{3.11}$$

we have that, for every $x \in \mathbb{R}^n$,

$$(\mathcal{C}_t f)(x) = \int_{\mathbb{R}^n} f(y) d\mathbf{p}_{t;x}(y) = \int_{\mathbb{R}^n} f(x+y) d\mathbf{p}_t(y), \quad \mathbf{p}_t \equiv \mathbf{p}_{t;0}. \tag{3.12}$$

Hence, the probability measure $\mathbf{p}_{t;x}$ is the x -translate of \mathbf{p}_t . It is also clear that, in this case, in formula (3.9) the functions a^{jk} and b^j , $j, k = 1, \dots, n$, must be constant.

Let now $\{\mathcal{C}_t\}_{t \in \mathbb{R}^+}$ be a Markovian semigroup in the Banach space $\mathcal{C}(\mathbb{R}^n)$ that commutes with translations. It can be shown that the infinitesimal generator \mathfrak{A} of such a semigroup verifies

$$\text{Dom}(\mathfrak{A}) \supset \mathcal{C}^2(\mathbb{R}^n). \tag{3.13}$$

Therefore, in this case, condition (3.7) is automatically satisfied. If, in addition, for all $\epsilon > 0$, $\mathbf{p}_t(\{y : |y| \geq \epsilon\}) = o(t)$ ($\mathbf{p}_t \equiv \mathbf{p}_{t;0}$), for $t \downarrow 0$, then condition (3.8) is satisfied too (as $\mathbf{p}_{t;x}$ is the x -translate of \mathbf{p}_t), and equation (3.9) holds, in this case with the real-valued functions a^{jk} and b^j , $j, k = 1, \dots, n$, that are actually constant (and the matrix $[a^{jk}]_{j,k=1}^n$ positive). We stress that, in the present paper, we are interested in the case where $\mathbf{p}_t(\infty) = 0$, for all $t > 0$ ('no masses escaping to infinity').

Let $\{\mathfrak{P}_t\}_{t \in \mathbb{R}^+}$ be a translation-invariant Markovian semigroup in $\mathcal{C}_0(\mathbb{R}^n)$, and let $\{\mu_t\}_{t \in \mathbb{R}^+}$ be the continuous convolution semigroup of measures that generates this semigroup. Then, extending the measure μ_t to a probability measure \mathbf{p}_t on \mathbb{R}^n ($\mathbf{p}_t(\infty) = 0$), one can define a Markovian semigroup $\{\mathcal{C}_t\}_{t \in \mathbb{R}^+}$ in $\mathcal{C}(\mathbb{R}^n)$ that commutes with translations:

$$(\mathcal{C}_t f)(x) := \int_{\mathbb{R}^n} f(x+y) d\mathbf{p}_t(y), \quad f \in \mathcal{C}(\mathbb{R}^n). \tag{3.14}$$

Assume, moreover, that $\{\mu_t\}_{t \in \mathbb{R}^+}$ satisfies (3.2), so that condition (3.8) is satisfied for the semigroup $\{\mathcal{C}_t\}_{t \in \mathbb{R}^+}$ (as well as condition (3.7)). $\mathcal{C}_0(\mathbb{R}^n)$ being an invariant subspace for the Markovian semigroup $\{\mathcal{C}_t\}_{t \in \mathbb{R}^+}$, we can define the linear operator $\mathfrak{J} : \mathcal{C}_0(\mathbb{R}^n) \cap \text{Dom}(\mathfrak{A}) \ni f \mapsto \mathfrak{A}f \in \mathcal{C}_0(\mathbb{R}^n)$, which is precisely the infinitesimal generator of $\{\mathfrak{P}_t\}_{t \in \mathbb{R}^+}$. Thus, from our previous discussion it follows that

$$(\mathfrak{J}f)(x) = \sum_{j=1}^n b^j \frac{\partial}{\partial x^j} f(x) + \sum_{j,k=1}^n a^{jk} \frac{\partial^2}{\partial x^j \partial x^k} f(x), \quad \forall f \in \mathcal{C}_c^2(\mathbb{R}^n), \tag{3.15}$$

for some real constants b_1, \dots, b_n and a positive matrix $[a^{jk}]_{j,k=1}^n$. It can be shown, moreover, that \mathfrak{J} is uniquely determined by (3.15). Clearly, the Lévy–Kintchine formula outlined in section 2 applies to the translation-invariant Markovian semigroup $\{\mathfrak{P}_t\}_{t \in \mathbb{R}^+}$ (with $G = \mathbb{R}^n$, of course), and the hypothesis that for all $\epsilon > 0$, $\mathbf{p}_t(\{y : |y| \geq \epsilon\}) = o(t)$, for $t \downarrow 0$, implies that the Lévy measure η appearing in (2.21) is identically zero (as a consequence of relation (2.22)). Therefore, formula (3.15) is coherent with the Lévy–Kintchine formula (2.20) (with $\mathfrak{A} \equiv 0$).

Finally, what we have recalled about the one-dimensional Brownian motion is easily recovered as a particular case. Let $\{\mathfrak{P}_t\}_{t \in \mathbb{R}^+}$ be a translation-invariant Markovian semigroup in $\mathcal{C}_0(\mathbb{R})$ such that the associated convolution semigroup of measures $\{\mu_t\}_{t \in \mathbb{R}^+}$ satisfies (3.2). Then, its infinitesimal generator \mathfrak{J} is uniquely determined by

$$(\mathfrak{J}f)(x) = b \frac{\partial}{\partial x} f(x) + a \frac{\partial^2}{\partial x^2} f(x), \quad \forall f \in \mathcal{C}_c^2(\mathbb{R}), \tag{3.16}$$

for some $a, b \in \mathbb{R}$, $a \geq 0$. If $a > 0$, for every $f \in \mathcal{C}(\mathbb{R})$, we have that

$$(\mathfrak{P}_t f)(x) = \int_{\mathbb{R}} f(y) \varrho_t(y-x) dy, \quad t > 0, \tag{3.17}$$

where $\wp_{(\cdot)}(\cdot) : \mathbb{R}_*^+ \times \mathbb{R} \rightarrow \mathbb{R}$ is the well-known fundamental solution of the drift-diffusion equation⁸

$$\frac{\partial}{\partial t} \wp_t(y) = -b \frac{\partial}{\partial y} \wp_t(y) + a \frac{\partial^2}{\partial y^2} \wp_t(y), \quad t > 0, \quad a > 0, \quad b \in \mathbb{R}. \quad (3.18)$$

On the other hand, for $a = 0$ we have a ‘pure drift regime’ and $\mu_t = \delta_{bt}$ (i.e. $|b|$ is the modulus of the drift velocity). Suppose, now, that the semigroup $\{\mathfrak{P}_t\}_{t \in \mathbb{R}^+}$ commutes with the reflection $x \mapsto -x$ as well. Then, it follows that $b = 0$. Moreover, if $a > 0$ (standard Brownian regime), the probability measure μ_t , for $t > 0$, is absolutely continuous with respect to the Lebesgue measure on \mathbb{R} and the Radon–Nikodým derivative \wp_t of μ_t with respect to this measure satisfies the diffusion equation (3.5), with $D = a$. Otherwise ($a = 0$), $\mathfrak{A} = 0$ and $\mu_t = \delta$, for all $t \in \mathbb{R}^+$.

4. Twirling superoperators and twirling semigroups

In sections 2 and 3, we have recalled the notion of left-invariant Markovian semigroup of operators in the Banach space $C_0(G)$, with G denoting a l.c.s.c. group, and we have illustrated this notion in the remarkable case where $G = \mathbb{R}^n$. In this section, we consider a class of semigroups of operators that is the central object of this paper. More precisely, we deal with semigroups of ‘superoperators’ acting in Banach spaces of operators. The most evident link between the two mentioned classes of operator semigroups is given by the fact that both are defined by means of convolution semigroups of probability measures on groups.

For the sake of clarity, we will establish the following notation. Given a (separable complex) Hilbert space \mathcal{H} , we will denote by \hat{B} a generic linear operator belonging to the Banach space $\mathcal{B}(\mathcal{H})$ of bounded operators in \mathcal{H} . The symbols \hat{A} , \hat{S} will denote generic operators in $\mathcal{B}_1(\mathcal{H})$ —the Banach space of trace-class operators, endowed with the trace norm $\|\cdot\|_{\text{tr}}$ —and in the Hilbert–Schmidt space $\mathcal{B}_2(\mathcal{H})$ (endowed with the norm $\|\cdot\|_{\text{HS}}$ induced by the Hilbert–Schmidt scalar product), respectively. As is well known, $\mathcal{B}_1(\mathcal{H})$ and $\mathcal{B}_2(\mathcal{H})$ are two-sided ideals in $\mathcal{B}(\mathcal{H})$, and $\mathcal{B}_1(\mathcal{H}) \subset \mathcal{B}_2(\mathcal{H})$. The dual space of $\mathcal{B}_1(\mathcal{H})$ can be identified with $\mathcal{B}(\mathcal{H})$ via the pairing

$$\mathcal{B}(\mathcal{H}) \times \mathcal{B}_1(\mathcal{H}) \ni (\hat{B}, \hat{A}) \mapsto \text{tr}(\hat{B}\hat{A}) \in \mathbb{C}. \quad (4.1)$$

We will denote by $\mathcal{L}(\mathcal{H})$, $\mathcal{L}'(\mathcal{H})$ the Banach spaces of bounded (super) operators in $\mathcal{B}_1(\mathcal{H})$ and $\mathcal{B}(\mathcal{H})$, respectively.

Let G be a l.c.s.c. group, and let U be a projective representation of G in \mathcal{H} . The following facts will be very useful for our purposes. The map

$$\underline{U \vee U} : G \rightarrow \mathcal{U}(\mathcal{B}_2(\mathcal{H})) \quad (4.2)$$

—where $\mathcal{U}(\mathcal{B}_2(\mathcal{H}))$ is the unitary group of the Hilbert space $\mathcal{B}_2(\mathcal{H})$ —defined by

$$\underline{U \vee U}(g)\hat{S} := U(g)\hat{S}U(g)^*, \quad \forall g \in G, \quad \forall \hat{S} \in \mathcal{B}_2(\mathcal{H}), \quad (4.3)$$

is a strongly continuous *unitary* representation, even in the case where the representation U is genuinely *projective*, see [36]. Clearly, for every $g \in G$ the unitary operator $\underline{U \vee U}(g)$ in $\mathcal{B}_2(\mathcal{H})$ induces the Banach space isomorphism (a surjective isometry) $\mathcal{B}_1(\mathcal{H}) \ni \hat{A} \mapsto \underline{U \vee U}(g)\hat{A} \in \mathcal{B}_1(\mathcal{H})$. Therefore, we can define the isometric representation

$$\begin{aligned} U \vee U : G &\rightarrow \mathcal{L}(\mathcal{H}), & U \vee U(g)\hat{A} &:= U(g)\hat{A}U(g)^*, \\ \forall g \in G, & \forall \hat{A} \in \mathcal{B}_1(\mathcal{H}), \end{aligned} \quad (4.4)$$

⁸ Namely $\wp_t(y) = \frac{1}{\sqrt{4\pi at}} \exp(-(y - bt)^2/4at)$ for $t > 0$.

keeping in mind the fact that $U \vee U(g)\hat{A} = \underline{U \vee U}(g)\hat{A}$ for all $\hat{A} \in \mathcal{B}_1(\mathcal{H})$ and $g \in G$.

Proposition 4.1. *The isometric representation $U \vee U$ of the l.c.s.c. group G in the Banach space $\mathcal{B}_1(\mathcal{H})$ is strongly continuous.*

Proof. Since G is a second countable (*a fortiori*, first countable) topological space, it is sufficient to show that $U \vee U$ is sequentially continuous. Let $\{g_n\}_{n \in \mathbb{N}}$ be a sequence in G converging to g . Then, for every $\hat{A} \in \mathcal{B}_1(\mathcal{H})$, the sequences

$$\{U \vee U(g_n)\hat{A} = \underline{U \vee U}(g_n)\hat{A}\}_{n \in \mathbb{N}}, \quad \{(U \vee U(g_n)\hat{A})^* = \underline{U \vee U}(g_n)\hat{A}^*\}_{n \in \mathbb{N}} \quad (4.5)$$

converge to $U \vee U(g)\hat{A}$ and $U \vee U(g)\hat{A}^*$, respectively, with respect to the Hilbert–Schmidt norm (the unitary representation $\underline{U \vee U}$ is strongly continuous), hence, with respect to the strong operator topology in $\mathcal{B}(\mathcal{H})$. Applying ‘Grümm’s convergence theorem’ (see [37], chapter 2), by this fact and by the fact that the representation $U \vee U$ is isometric, we find out that the sequence $\{U \vee U(g_n)\hat{A}\}_{n \in \mathbb{N}}$ converges to $U \vee U(g)\hat{A}$ with respect to the trace norm, as well. \square

Next, observe that, for every $\hat{B} \in \mathcal{B}(\mathcal{H})$, the map $G \ni g \mapsto U(g)^*\hat{B}U(g) \in \mathcal{B}(\mathcal{H})$ is weakly continuous (since $\langle \phi, U(g)^*\hat{B}U(g)\psi \rangle = \text{tr}(\hat{B}(U \vee U(g)|\psi\rangle\langle\phi|))$, for all $\phi, \psi \in \mathcal{H}$, and the representation $U \vee U$ is strongly continuous). Then, given a *finite* Borel measure μ on G , one can consider the bounded linear map $\mathfrak{D}_\mu^U : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ defined by

$$\mathfrak{D}_\mu^U \hat{B} := \int_G d\mu(g) U(g)^*\hat{B}U(g), \quad \hat{B} \in \mathcal{B}(\mathcal{H}), \quad (4.6)$$

where on the rhs of (4.6) a weak integral (i.e. an integral converging with respect to the weak operator topology in $\mathcal{B}(\mathcal{H})$) is understood. In the case where μ is normalized ($\mu(G) = 1$, i.e. μ is a probability measure), it is obvious that $\mathfrak{D}_\mu^U I = I$ and it is easy to check that the linear map \mathfrak{D}_μ^U is a *contraction* (i.e. its norm is not larger than one). From this point onward, we will assume that μ belongs to $\mathcal{M}^1(G)$.

It is clear that the map \mathfrak{D}_μ^U is positive. One can prove, moreover, that it is *completely* positive. In fact, recalling the necessary and sufficient condition (2.25), for every $m \in \mathbb{N}$ the positivity of the function $M : G \rightarrow \mathbb{R}$,

$$M(g) := \sum_{j,k=1}^m \langle \psi_j, U(g)^*\hat{B}_j^*\hat{B}_k U(g)\psi_k \rangle, \quad g \in G, \quad (4.7)$$

for any m -tuple $\{\psi_1, \dots, \psi_m\}$ in \mathcal{H} and any m -tuple $\{\hat{B}_1, \dots, \hat{B}_m\}$ in $\mathcal{B}(\mathcal{H})$, implies that

$$\sum_{j,k=1}^m \langle \psi_j, \mathfrak{D}_\mu^U(\hat{B}_j^*\hat{B}_k)\psi_k \rangle = \int_G d\mu(g) M(g) \geq 0. \quad (4.8)$$

One can show that the map \mathfrak{D}_μ^U is the adjoint—with respect to the pairing (4.1)—of the linear application $\mathfrak{S}_\mu^U : \mathcal{B}_1(\mathcal{H}) \rightarrow \mathcal{B}_1(\mathcal{H})$ defined by

$$\mathfrak{S}_\mu^U \hat{A} := \int_G d\mu(g) (U \vee U(g)\hat{A}), \quad \hat{A} \in \mathcal{B}_1(\mathcal{H}), \quad (4.9)$$

where, again, a weak integral (weak operator topology in $\mathcal{B}(\mathcal{H})$) is understood. Observe, in fact, that $\mathfrak{S}_\mu^U \hat{A}$ is a bounded operator (and $\hat{A} \geq 0 \Rightarrow \mathfrak{S}_\mu^U \hat{A} \geq 0$); moreover, it is in the trace class and

$$\text{tr}(\mathfrak{S}_\mu^U \hat{A}) = \text{tr}(\hat{A}), \quad \forall \hat{A} \in \mathcal{B}_1(\mathcal{H}). \quad (4.10)$$

This last assertion is verified assuming—without loss of generality, since $\hat{A} \in \mathcal{B}_1(\mathcal{H})$ can be expressed as a linear combination of four *positive* trace-class operators, namely $\hat{A} = \hat{A}_1 - \hat{A}_2 + i(\hat{A}_3 - \hat{A}_4)$ —that \hat{A} is positive, and using the definition of the trace and the ‘monotone convergence theorem’ for permuting the possibly infinite sum (associated with the trace) with the integral on G . Next, one can verify that

$$\text{tr}(\hat{B}(\mathfrak{S}_\mu^U \hat{A})) = \text{tr}((\mathfrak{D}_\mu^U \hat{B})\hat{A}), \quad \forall \hat{A} \in \mathcal{B}_1(\mathcal{H}), \quad \forall \hat{B} \in \mathcal{B}(\mathcal{H}). \quad (4.11)$$

To this aim, assume—again, without loss of generality—that $\hat{A} \in \mathcal{B}_1(\mathcal{H})$ and $\hat{B} \in \mathcal{B}(\mathcal{H})$ are both positive. Then, given an orthonormal basis $\{\psi_l\}_{l \in \mathbb{N}}$ in \mathcal{H} ($\mathcal{N} \subset \mathbb{N}$), we have

$$\begin{aligned} \text{tr}(\hat{B}(\mathfrak{S}_\mu^U \hat{A})) &= \text{tr}(\hat{B}^{1/2}(\mathfrak{S}_\mu^U \hat{A})\hat{B}^{1/2}) \\ &= \sum_{l \in \mathbb{N}} \int_G d\mu(g) \langle \psi_l, \hat{B}^{1/2} U(g) \hat{A} U(g)^* \hat{B}^{1/2} \psi_l \rangle. \end{aligned} \quad (4.12)$$

At this point, since the integrand function on the rhs of (4.12) is positive, we can apply the ‘monotone convergence theorem’ and permute the (possibly infinite) sum with the integral, thus getting

$$\begin{aligned} \text{tr}(\hat{B}(\mathfrak{S}_\mu^U \hat{A})) &= \int_G d\mu(g) \text{tr}(\hat{B}^{1/2} U(g) \hat{A} U(g)^* \hat{B}^{1/2}) \\ &= \int_G d\mu(g) \text{tr}(\hat{A}^{1/2} U(g)^* \hat{B} U(g) \hat{A}^{1/2}) \\ &= \int_G d\mu(g) \sum_{l \in \mathbb{N}} \langle \psi_l, \hat{A}^{1/2} U(g)^* \hat{B} U(g) \hat{A}^{1/2} \psi_l \rangle. \end{aligned} \quad (4.13)$$

Eventually, we can again permute the sum with the integral and obtain relation (4.11). Note that the first line of (4.13) implies that \mathfrak{S}_μ^U coincides with the weak integral—i.e. the integral with respect to the weak topology of bounded operators in $\mathcal{B}_1(\mathcal{H})$ — $\int_G d\mu(g) U \vee U(g)$. Also note that, since \mathfrak{D}_μ^U is a contraction in $\mathcal{B}(\mathcal{H})$, \mathfrak{S}_μ^U is a contraction in $\mathcal{B}_1(\mathcal{H})$; indeed

$$\begin{aligned} \|\mathfrak{S}_\mu^U \hat{A}\|_{\text{tr}} &= \sup \{ |\text{tr}(\hat{B}(\mathfrak{S}_\mu^U \hat{A}))| : \hat{B} \in \mathcal{B}(\mathcal{H}), \|\hat{B}\| = 1 \} \\ &= \sup \{ |\text{tr}((\mathfrak{D}_\mu^U \hat{B})\hat{A})| : \hat{B} \in \mathcal{B}(\mathcal{H}), \|\hat{B}\| = 1 \} \\ &\leq \|\hat{A}\|_{\text{tr}} \sup \{ \|\mathfrak{D}_\mu^U \hat{B}\| : \hat{B} \in \mathcal{B}(\mathcal{H}), \|\hat{B}\| = 1 \} \leq \|\hat{A}\|_{\text{tr}} \end{aligned} \quad (4.14)$$

for all $\hat{A} \in \mathcal{B}_1(\mathcal{H})$.

We can summarize our previous discussion by stating the following result.

Proposition 4.2. *For every projective representation U of a l.c.s.c group G in \mathcal{H} and for every probability measure μ on G , the bounded linear map $\mathfrak{S}_\mu^U : \mathcal{B}_1(\mathcal{H}) \rightarrow \mathcal{B}_1(\mathcal{H})$ defined by (4.9) is a contraction, and it is positive and trace preserving. Moreover, we have the formula*

$$\mathfrak{S}_\mu^U = \int_G d\mu(g) U \vee U(g), \quad (4.15)$$

where the integral holds in the weak sense. The bounded linear map $\mathfrak{D}_\mu^U : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ defined by (4.6) is the adjoint of \mathfrak{S}_μ^U . It is a completely positive map.

Remark 4.1. Suppose that the Hilbert space of the representation U is finite dimensional. Then, for every probability measure μ on G , \mathfrak{S}_μ^U is a completely positive, trace-preserving linear map which is also *unital*, i.e. such that $\mathfrak{S}_\mu^U I = I$. Therefore, it is a *bistochastic* (or ‘doubly stochastic’) linear map [21]. Clearly, the bistochastic linear maps in $\mathcal{L}(\mathcal{H})$ form a convex set. The determination of the extreme points of this convex set is an interesting

problem [38]. From the physicist’s point of view, these maps are characterized by the property of leaving the maximally mixed state invariant.

In the case where G is a unitary group ($U(n)$ or $SU(n)$), μ is the Haar measure on G (normalized in such a way that $\mu(G) = 1$) and U is the defining representation of G , we have that \mathfrak{S}_μ^U is the ‘standard’ *twirling superoperator* (in $\mathcal{B}(\mathbb{C}^n)$). Therefore, in a general case, it is quite natural to extend this terminology and call \mathfrak{S}_μ^U the (U, μ) -*twirling superoperator*; the map \mathfrak{D}_μ^U will be called, accordingly, the *dual (U, μ) -twirling superoperator*. Since any convex combination of two probability measures on G is again a probability measure, the following result holds.

Proposition 4.3. *For every projective representation $U : G \rightarrow U(\mathcal{H})$, the subsets*

$$\{\mathfrak{S}_\mu^U : \mu \in \mathcal{M}^1(G)\}, \quad \{\mathfrak{D}_\mu^U : \mu \in \mathcal{M}^1(G)\} \tag{4.16}$$

of the Banach spaces $\mathcal{L}(\mathcal{H})$ and $\mathcal{L}'(\mathcal{H})$, respectively, are convex.

Remark 4.2. It is worth observing that in definition (4.9) of the twirling superoperator one may replace the weak integral with a Bochner integral (relative to the Banach space $\mathcal{B}_1(\mathcal{H})$).

It is also an interesting fact that a probability measure μ on G allows us to define a bounded linear map $\check{\mathfrak{S}}_\mu^U : \mathcal{B}_2(\mathcal{H}) \rightarrow \mathcal{B}_2(\mathcal{H})$ along the scheme already outlined for the maps \mathfrak{D}_μ^U and \mathfrak{S}_μ^U , i.e.

$$\check{\mathfrak{S}}_\mu^U \hat{S} := \int_G d\mu(g) (U \vee U(g) \hat{S}), \quad \hat{S} \in \mathcal{B}_2(\mathcal{H}), \tag{4.17}$$

where, once again, one can show that the map $\check{\mathfrak{S}}_\mu^U$ is well defined (with the integral on the rhs of (4.17) regarded, equivalently, as a weak or as a Bochner integral). Indeed, observe that, for every $\hat{S} \in \mathcal{B}_2(\mathcal{H})$, we have

$$\begin{aligned} 0 &< \sum_{l \in \mathcal{N}} \int_G d\mu(g) \int_G d\mu(h) \langle \psi_l, U(g) \hat{S}^* U(g)^* U(h) \hat{S} U(h)^* \psi_l \rangle \\ &\leq \sum_{l \in \mathcal{N}} \int_G d\mu(g) \int_G d\mu(h) |\langle \psi_l, U(g) \hat{S}^* U(g)^* U(h) \hat{S} U(h)^* \psi_l \rangle| \\ &\leq \int_G d\mu(g) \int_G d\mu(h) \sum_{l \in \mathcal{N}} |\langle \psi_l, U(g) \hat{S}^* U(g)^* U(h) \hat{S} U(h)^* \psi_l \rangle| \\ &\leq \int_G d\mu(g) \int_G d\mu(h) \|U(g) \hat{S}^* U(g)^* U(h) \hat{S} U(h)^*\|_{\text{tr}} \leq \|\hat{S}\|_{\text{HS}}^2. \end{aligned} \tag{4.18}$$

The previous argument also shows that $\check{\mathfrak{S}}_\mu^U$ is a contraction. It is clear, moreover, that the map \mathfrak{S}_μ^U can be regarded as the restriction to the trace-class operators of the map $\check{\mathfrak{S}}_\mu^U$.

From definition (4.9) it is clear that the map

$$\mathcal{M}^1(G) \ni \mu \mapsto \mathfrak{S}_\mu^U \in \text{DM}(\mathcal{H}) \tag{4.19}$$

is a homomorphism of the semigroup $\mathcal{M}^1(G)$ —with respect to convolution—into the semigroup $\text{DM}(\mathcal{H})$ —with respect to composition—of (quantum) *dynamical maps* in $\mathcal{B}_1(\mathcal{H})$, namely, of the semigroup consisting of all positive, trace-preserving, bounded linear maps in $\mathcal{B}_1(\mathcal{H})$, whose adjoints (acting in the Banach space $\mathcal{B}(\mathcal{H})$) are completely positive [14]. This observation leads us to consider an interesting class of (continuous) one-parameter semigroups of superoperators.

Indeed—given a continuous one-parameter convolution semigroup $\{\mu_t\}_{t \in \mathbb{R}^+} \subset \mathcal{M}^1(G)$ of measures on G and a projective representation U of G in \mathcal{H} —for every $t \geq 0$, we can as above define the (U, μ_t) -twirling superoperator:

$$\mathfrak{S}_t \equiv \mathfrak{S}_{\mu_t}^U : \mathcal{B}_1(\mathcal{H}) \rightarrow \mathcal{B}_1(\mathcal{H}), \quad t \geq 0, \quad (\mathfrak{S}_0 = I). \quad (4.20)$$

The fact that $\{\mathfrak{S}_t\}_{t \in \mathbb{R}^+}$ enjoys the one-parameter semigroup property is a consequence of the fact that $\{\mu_t\}_{t \in \mathbb{R}^+}$ is a convolution semigroup and the map (4.19) is a homomorphism. Moreover, the semigroup $\{\mathfrak{S}_t\}_{t \in \mathbb{R}^+}$ is strongly right continuous at $t = 0$. This is a consequence of the continuity of $\{\mu_t\}_{t \in \mathbb{R}^+}$ and of proposition 4.1. Actually, as recalled in section 2, it suffices to prove the weak right continuity at $t = 0$ of the semigroup $\{\mathfrak{S}_t\}_{t \in \mathbb{R}^+}$. To this aim, observe that, for every $\hat{A} \in \mathcal{B}_1(\mathcal{H})$ and $\hat{B} \in \mathcal{B}(\mathcal{H})$, the function

$$G \ni g \mapsto \text{tr}(\hat{B}(U \vee U(g)\hat{A})) \in \mathbb{C} \quad (4.21)$$

is continuous (equivalently, the representation $U \vee U$ is weakly continuous). Also note that $|\text{tr}(\hat{B}(U \vee U(g)\hat{A})) - \text{tr}(\hat{B}\hat{A})| \leq \|\hat{B}(U \vee U(g)\hat{A})\|_{\text{tr}} + \|\hat{B}\hat{A}\|_{\text{tr}} \leq 2\|\hat{B}\|\|\hat{A}\|_{\text{tr}}$, (4.22) for all $g \in G$. Therefore, the function

$$G \ni g \mapsto |\text{tr}(\hat{B}(U \vee U(g)\hat{A})) - \text{tr}(\hat{B}\hat{A})| \in \mathbb{R} \quad (4.23)$$

is bounded and continuous. At this point, we can exploit the fact that $\lim_{t \downarrow 0} \mu_t = \delta$ (weakly). By this relation, since

$$\begin{aligned} |\text{tr}(\hat{B}(\mathfrak{S}_t\hat{A})) - \text{tr}(\hat{B}\hat{A})| &= \left| \int_G d\mu_t(g) (\text{tr}(\hat{B}(U \vee U(g)\hat{A})) - \text{tr}(\hat{B}\hat{A})) \right| \\ &\leq \int_G d\mu_t(g) |\text{tr}(\hat{B}(U \vee U(g)\hat{A})) - \text{tr}(\hat{B}\hat{A})|, \end{aligned} \quad (4.24)$$

we conclude that

$$\lim_{t \downarrow 0} |\text{tr}(\hat{B}(\mathfrak{S}_t\hat{A})) - \text{tr}(\hat{B}\hat{A})| = 0, \quad \forall \hat{A} \in \mathcal{B}_1(\mathcal{H}), \quad \forall \hat{B} \in \mathcal{B}(\mathcal{H}). \quad (4.25)$$

This completes the proof of the continuity of the one-parameter semigroup $\{\mathfrak{S}_t\}_{t \in \mathbb{R}^+}$.

At this point, recalling that a *quantum dynamical semigroup* [14] in $\mathcal{B}_1(\mathcal{H})$ is a (strongly continuous) one-parameter semigroup of quantum dynamical maps in $\mathcal{B}_1(\mathcal{H})$, we can resume our preceding discussion stating the following result.

Proposition 4.4. *The contraction semigroup $\{\mathfrak{S}_t : \mathcal{B}_1(\mathcal{H}) \rightarrow \mathcal{B}_1(\mathcal{H})\}_{t \in \mathbb{R}^+}$ is a quantum dynamical semigroup.*

Remark 4.3. Recalling remark 4.1, we have that—in the case where the Hilbert space of the representation U is finite dimensional—the dynamical semigroup $\{\mathfrak{S}_t\}_{t \in \mathbb{R}^+}$ is a *bistochastic dynamical semigroup*. A complete characterization of the twirling semigroups associated with finite-dimensional representations of Lie groups will be provided in section 5.

Remark 4.4. The contraction $\check{\mathfrak{S}}_\mu^U$ defined by (4.17) allows us to define, for every continuous convolution semigroup $\{\mu_t\}_{t \in \mathbb{R}^+}$ of probability measures on G , a contraction semigroup $\{\check{\mathfrak{S}}_t\}_{t \in \mathbb{R}^+}$ in the Hilbert space $\mathcal{B}_2(\mathcal{H})$, i.e.

$$\check{\mathfrak{S}}_t \equiv \check{\mathfrak{S}}_{\mu_t}^U : \mathcal{B}_2(\mathcal{H}) \rightarrow \mathcal{B}_2(\mathcal{H}). \quad (4.26)$$

The fact that $w\text{-}\lim_{t \downarrow 0} \check{\mathfrak{S}}_t = I$ can be proved by means of a procedure analogous to that adopted for the semigroup $\{\mathfrak{S}_t\}_{t \in \mathbb{R}^+}$.

In the following, we will call $\{\mathfrak{S}_t\}_{t \in \mathbb{R}^+}$ the *twirling semigroup* associated with (or induced by) the pair $(U, \{\mu_t\}_{t \in \mathbb{R}^+})$. We stress that, in general, a twirling semigroup will be induced by different pairs of the type (projective representation, convolution semigroup of measures).

5. Brownian motion on Lie groups and open quantum systems

In this section, we will study the twirling semigroups of operators induced by representations of *Lie groups*. This is a particularly interesting case because the differential structure of a Lie group allows us to obtain a characterization of the infinitesimal generators of the associated twirling semigroups. The main technical tool will be the Lévy–Kintchine formula (2.20). In order to avoid all mathematical intricacies related to infinite-dimensional Hilbert spaces, we will consider the case where the group representations involved are finite dimensional, case which is relevant, for instance, in applications to quantum computation [18]. The general case will be considered elsewhere.

Thus, in the following we will deal with a smooth, finite-dimensional unitary representation U of a Lie group G (of dimension n) in a N -dimensional (complex) Hilbert space \mathcal{H} . It is clear that, in this case, $\mathcal{B}(\mathcal{H}) = \mathcal{B}_1(\mathcal{H}) = \mathcal{B}_2(\mathcal{H})$ and $\mathcal{L}(\mathcal{H}) = \mathcal{L}'(\mathcal{H})$. Since all norms in $\mathcal{B}(\mathcal{H})$ (or $\mathcal{L}(\mathcal{H})$) induce the same topology (as \mathcal{H} is finite dimensional), all our statements involving topological properties of $\mathcal{B}(\mathcal{H})$ (or $\mathcal{L}(\mathcal{H})$)—convergence, continuity, compactness, etc—are to be understood as referred to this topology. We will denote, as usual, by $\mathcal{U}(\mathcal{H})$ the unitary group of \mathcal{H} , endowed with the topology inherited from $\mathcal{B}(\mathcal{H})$; it is well known that $\mathcal{U}(\mathcal{H})$ is compact with respect to this topology. Let us fix once and for all a basis $\{\xi_1, \dots, \xi_n\}$ in the Lie algebra $\text{Lie}(G)$ and a system of adapted coordinates $\{g \mapsto \bar{x}^1(g), \dots, g \mapsto \bar{x}^n(g)\}$ based at the identity. We will use the notations adopted in section 2, usually with no further explanation.

Remark 5.1. We will repeatedly use the following fact. Let $f : G \rightarrow \mathcal{L}(\mathcal{H})$ be a bounded continuous function. Then, for every probability measure μ on G , $\mu(f) := \int_G f(g) d\mu(g)$ belongs to the closure of the convex hull $\text{co}(f(G)) \subset \mathcal{L}(\mathcal{H})$. Indeed, observe that G is (homeomorphic to) a separable metric space. Then, there exists a sequence $\{\mu_m\}_{m \in \mathbb{N}}$ of *finitely supported* probability measures on G weakly converging to μ (see [39], chapter 2, theorem 6.3). Hence, $\mu_m(f) \in \text{co}(f(G))$ and $\mu(f) = \lim_{m \rightarrow \infty} \mu_m(f) \in \overline{\text{co}}(f(G))$.

We have observed in section 4 that a twirling superoperator is a bistochastic linear map, see remark 4.1. We will now show that, actually, it belongs to a special class of bistochastic maps, namely the class of ‘random unitary maps’.

Definition 5.1. A quantum dynamical map $\mathfrak{U} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ is said to be a random unitary map if it admits a decomposition of the form

$$\mathfrak{U}\hat{A} = \sum_{k=1}^{\mathcal{N}} p_k V_k \hat{A} V_k^*, \quad \mathcal{N} \in \mathbb{N}, \tag{5.1}$$

where $\{V_k\}_{k=1}^{\mathcal{N}}$ is a set of unitary operators in \mathcal{H} and $\{p_k\}_{k=1}^{\mathcal{N}} \subset \mathbb{R}_*^+$ is a probability distribution; i.e. if it is a convex combination of unitary transformations. The cardinality $\text{card}(\mathfrak{U})$ of a random unitary map \mathfrak{U} is the minimum number of terms required in a decomposition of \mathfrak{U} of the type (5.1).

Observe that the random unitary maps acting in $\mathcal{B}(\mathcal{H})$ form a semigroup $\text{DM}_{\text{ru}}(\mathcal{H})$ contained in the semigroup of quantum dynamical maps $\text{DM}(\mathcal{H})$. It is natural to consider the non-zero positive integer $\mathfrak{c}(\mathbb{N})$ defined as follows:

$$\mathfrak{c}(\mathbb{N}) := \sup\{\text{card}(\mathfrak{U}) \in \mathbb{N} : \mathfrak{U} \in \text{DM}_{\text{ru}}(\mathcal{H})\}, \quad \mathbb{N} = \dim(\mathcal{H}). \tag{5.2}$$

Since a random unitary map sends the subspace, formed by the traceless operators, of the real vector space $\mathcal{B}_{\mathbb{R}}(\mathcal{H})$ (of self-adjoint operators in \mathcal{H}) into itself, applying Carathéodory theorem

one finds the estimate $c(N) \leq (N^2 - 1)^2 + 1 = N^4 - 2N^2 + 2$. This estimate is not tight. For instance, in the case where $N = 2$, it is known that all bistochastic maps (hence, all random unitary maps) are ‘Pauli channels’ [21]; thus, $c(2) = 4$. To the best of our knowledge, the generic integer $c(N)$ is unknown, but stricter bounds for the cardinality of a random unitary map can be provided and it turns out that $c(N) \leq N^2$ [40].

Consider, now, a subgroup \mathcal{V} of the group $\mathcal{U}(\mathcal{H})$. The closure $\overline{\mathcal{V}}$ of \mathcal{V} is a subgroup of $\mathcal{U}(\mathcal{H})$, as well. Denote by $\text{DM}_{\text{ru}}(\mathcal{V})$ the subset of $\text{DM}_{\text{ru}}(\mathcal{H})$ formed by those superoperators of the form (5.1) with the set of unitary operators $\{V_k\}_{k=1}^N$ contained in \mathcal{V} . Clearly, $\text{DM}_{\text{ru}}(\mathcal{H}) = \text{DM}_{\text{ru}}(\mathcal{U}(\mathcal{H}))$, and $\text{DM}_{\text{ru}}(\mathcal{V})$ is a subsemigroup of $\text{DM}_{\text{ru}}(\mathcal{H})$. It is clear that, defining

$$\mathcal{V} \vee \mathcal{V} := \{V(\cdot)V^* \in \mathcal{L}(\mathcal{H}) : V \in \mathcal{V}\}, \tag{5.3}$$

the semigroup $\text{DM}_{\text{ru}}(\mathcal{V})$ is nothing but the convex hull of the set $\mathcal{V} \vee \mathcal{V}$:

$$\text{DM}_{\text{ru}}(\mathcal{V}) = \text{co}(\mathcal{V} \vee \mathcal{V}). \tag{5.4}$$

Lemma 5.1. *For every subgroup \mathcal{V} of $\mathcal{U}(\mathcal{H})$, the semigroup $\text{DM}_{\text{ru}}(\overline{\mathcal{V}})$ is a compact convex subset of $\mathcal{L}(\mathcal{H})$ that coincides with the set $\overline{\text{DM}_{\text{ru}}(\mathcal{V})}$. Thus, in particular, the semigroup $\text{DM}_{\text{ru}}(\mathcal{H})$ is a compact convex subset of $\mathcal{L}(\mathcal{H})$.*

Proof. Note that the map

$$\mathcal{U}(\mathcal{H}) \ni V \mapsto V(\cdot)V^* \in \mathcal{L}(\mathcal{H}) \tag{5.5}$$

is continuous. Hence, the image, through this map, of the closed subgroup $\overline{\mathcal{V}}$ of $\mathcal{U}(\mathcal{H})$ —i.e. $\overline{\mathcal{V}} \vee \overline{\mathcal{V}}$ —is a compact subset \mathcal{K} of $\mathcal{L}(\mathcal{H})$. Recall that, in a finite-dimensional (real or complex) vector space, the convex hull of a compact set is compact, and the closure of the convex hull of a bounded set coincides with the convex hull of the closure of this set. Then, $\text{DM}_{\text{ru}}(\overline{\mathcal{V}}) = \text{co}(\overline{\mathcal{V}} \vee \overline{\mathcal{V}})$ is a compact subset of $\mathcal{L}(\mathcal{H})$. Moreover, $\text{co}(\overline{\mathcal{V}} \vee \overline{\mathcal{V}})$ coincides with the closure $\overline{\text{co}(\mathcal{V} \vee \mathcal{V})} = \overline{\text{DM}_{\text{ru}}(\mathcal{V})}$ of $\text{co}(\mathcal{V} \vee \mathcal{V})$. Indeed, $\overline{\mathcal{V}} \vee \overline{\mathcal{V}} = \overline{\mathcal{V} \vee \mathcal{V}}$ (as the map (5.5) is continuous, $\overline{\mathcal{V}} \vee \overline{\mathcal{V}} \subset \overline{\mathcal{V} \vee \mathcal{V}}$ and $\overline{\mathcal{V}} \vee \overline{\mathcal{V}} = \overline{\mathcal{V} \vee \mathcal{V}} \supset \overline{\mathcal{V} \vee \mathcal{V}}$); hence $\text{co}(\overline{\mathcal{V}} \vee \overline{\mathcal{V}}) = \text{co}(\overline{\mathcal{V} \vee \mathcal{V}}) = \overline{\text{co}(\mathcal{V} \vee \mathcal{V})}$. \square

Definition 5.2. *A random unitary semigroup acting in $\mathcal{B}(\mathcal{H})$ is a quantum dynamical semigroup taking values in the semigroup $\text{DM}_{\text{ru}}(\mathcal{H})$.*

Proposition 5.1. *Every twirling superoperator in $\mathcal{B}(\mathcal{H})$ is a random unitary map. Therefore, every twirling semigroup acting in $\mathcal{B}(\mathcal{H})$ is a random unitary semigroup.*

Proof. Expression (4.15) of a twirling superoperator involves an integral that, in the case where \mathcal{H} is finite dimensional, can be considered to be defined with respect to the topology of $\mathcal{L}(\mathcal{H})$. Thus, taking into account remark 5.1, from lemma 5.1 the statement follows. \square

A quantum dynamical semigroup $\{\Omega_t : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})\}_{t \in \mathbb{R}^+}$ is completely characterized by its (in this case, of course, bounded) infinitesimal generator \mathcal{L} :

$$\mathcal{L} = \lim_{t \downarrow 0} t^{-1}(\Omega_t - I). \tag{5.6}$$

According to the Gorini–Kossakowski–Lindblad–Sudarshan classification theorem [12, 13], \mathcal{L} has the general form

$$\mathcal{L} \hat{A} = -i[\hat{H}, \hat{A}] + \mathfrak{F} \hat{A} - \frac{1}{2}((\mathfrak{F}^* I) \hat{A} + \hat{A} (\mathfrak{F}^* I)), \tag{5.7}$$

where \hat{H} is a traceless self-adjoint operator in \mathcal{H} , $\mathfrak{F} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ a completely positive map and \mathfrak{F}^* its adjoint with respect to the Hilbert–Schmidt scalar product in $\mathcal{B}(\mathcal{H})$.

Remark 5.2. As is well known [21], a completely positive map $\mathfrak{K} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ can be expressed in the *Kraus–Stinespring–Sudarshan canonical form*:

$$\mathfrak{K}(\hat{A}) = \sum_{k=1}^{N^2} \gamma_k \hat{K}_k \hat{A} \hat{K}_k^*, \quad \gamma_k \geq 0, \quad \hat{A} \in \mathcal{B}(\mathcal{H}), \quad (5.8)$$

where $\hat{K}_1, \dots, \hat{K}_{N^2}$ are linear operators in \mathcal{H} such that

$$\langle \hat{K}_j, \hat{K}_k \rangle_{\text{HS}} := \text{tr}(\hat{K}_j^* \hat{K}_k) = \delta_{jk}, \quad j, k = 1, \dots, N^2. \quad (5.9)$$

However, it can be easily shown that the completely positive map \mathfrak{F} in formula (5.7) can be assumed, without loss of generality, to be of the form

$$\mathfrak{F} \hat{A} = \sum_{k=1}^{N^2-1} \gamma_k \hat{F}_k \hat{A} \hat{F}_k^*, \quad \gamma_k \geq 0, \quad \left(\mathfrak{F}^* \hat{A} = \sum_{k=1}^{N^2-1} \gamma_k \hat{F}_k^* \hat{A} \hat{F}_k \right), \quad (5.10)$$

where the $N^2 - 1$ linear operators $\hat{F}_1, \dots, \hat{F}_{N^2-1}$ form an orthonormal basis—with respect to the Hilbert–Schmidt scalar product $\langle \cdot, \cdot \rangle_{\text{HS}}$ —in the orthogonal complement of the one-dimensional subspace of $\mathcal{B}(\mathcal{H})$ generated by the identity operator (thus, they are traceless). In this way, formula (5.7) gives the so-called diagonal form[11] of the infinitesimal generator \mathfrak{L} .

Later on, we will prove a generalization of a classical result of Kümmerer and Maassen [17]; see theorem 5.1 below. As a first step, from [17] we can extract some useful information on random unitary semigroups. Given a subgroup \mathcal{V} of the group $\mathcal{U}(\mathcal{H})$, we will denote by $\mathcal{C}(\mathcal{V})$ the closure of the convex cone in $\mathcal{L}(\mathcal{H})$ generated by the set $\mathcal{V} \vee \mathcal{V} - I$, namely

$$\mathcal{C}(\mathcal{V}) := \overline{\text{co-cone}}(\{(V(\cdot)V^* - I) \in \mathcal{L}(\mathcal{H}) : V \in \mathcal{V}\}). \quad (5.11)$$

In particular, we will adopt the shorthand notation $\mathcal{C}(\mathcal{H}) \equiv \mathcal{C}(\mathcal{U}(\mathcal{H}))$.

Proposition 5.2. *The following facts are equivalent.*

- (a) *The quantum dynamical semigroup $\{\mathfrak{Q}_t : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})\}_{t \in \mathbb{R}^+}$ is a random unitary semigroup.*
- (b) *The infinitesimal generator of the quantum dynamical semigroup $\{\mathfrak{Q}_t : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})\}_{t \in \mathbb{R}^+}$ belongs to the closed convex cone $\mathcal{C}(\mathcal{H})$.*
- (c) *The infinitesimal generator \mathfrak{L} of the quantum dynamical semigroup $\{\mathfrak{Q}_t : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})\}_{t \in \mathbb{R}^+}$ is of the form (5.7), with the completely positive map $\mathfrak{F} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ of the form*

$$\mathfrak{F} \hat{A} = \sum_{k=1}^K \hat{E}_k \hat{A} \hat{E}_k + \gamma_0 \mathfrak{U} \hat{A}, \quad \hat{E}_k \in \mathcal{B}_{\mathbb{R}}(\mathcal{H}), \quad \gamma_0 \geq 0, \quad \mathfrak{U} \in \text{DM}_{\text{ru}}(\mathcal{H}) \quad (5.12)$$

for all $\hat{A} \in \mathcal{B}(\mathcal{H})$.

- (d) *The infinitesimal generator \mathfrak{L} of the quantum dynamical semigroup $\{\mathfrak{Q}_t : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})\}_{t \in \mathbb{R}^+}$ is of the form*

$$\mathfrak{L} \hat{A} = -i[\hat{H}, \hat{A}] + \sum_{k=1}^{N^2-1} \gamma_k \left(\hat{L}_k \hat{A} \hat{L}_k - \frac{1}{2}(\hat{L}_k^2 \hat{A} + \hat{A} \hat{L}_k^2) \right) + \gamma_0(\mathfrak{U} - I)\hat{A}, \quad \hat{A} \in \mathcal{B}(\mathcal{H}), \quad (5.13)$$

where \hat{H} is a traceless self-adjoint operator, $\hat{L}_1, \dots, \hat{L}_{N^2-1}$ are the traceless self-adjoint operators such that

$$\langle \hat{L}_j, \hat{L}_k \rangle_{\text{HS}} = \delta_{jk}, \quad j, k = 1, \dots, N^2 - 1, \tag{5.14}$$

\mathfrak{U} is a random unitary map acting in $\mathcal{B}(\mathcal{H})$ and $\gamma_0, \dots, \gamma_{N^2-1}$ are non-negative numbers.

Proof. The equivalence of (a), (b) and (c) is proved in [17] (see theorem 1.1.1.; here we have only adapted terminology and results to our context). The equivalence of (c) and (d) is straightforward. Hint: in order to get (d) from (c), expand the self-adjoint operators $\{\hat{E}_k\}_{k=1}^K - \hat{E}_k = \sum_{l=0}^{N^2-1} c_{kl} \hat{F}_l$ —with respect to an orthonormal basis $\{\hat{F}_l\}_{l=0}^{N^2-1}$ in $\mathcal{B}_{\mathbb{R}}(\mathcal{H})$ ($\langle \hat{F}_j, \hat{F}_l \rangle_{\text{HS}} = \delta_{jl}$), with $\hat{F}_0 = I$; then, diagonalize the positive real matrix $[M_{lm}]_{l,m=1}^{N^2-1}$, where $M_{lm} = \sum_{k=1}^K c_{kl} c_{km}$, by means of an orthogonal transformation, and next use the orthogonal matrix involved in this transformation for defining a new orthonormal basis in the subspace of $\mathcal{B}_{\mathbb{R}}(\mathcal{H})$ formed by traceless operators. \square

For reasons that will be clear later on, it is convenient to single out a special class of random unitary semigroups, namely the *Gaussian dynamical semigroups*.

Definition 5.3. We will say that a quantum dynamical semigroup $\{\mathfrak{Q}_t\}_{t \in \mathbb{R}^+}$ acting in $\mathcal{B}(\mathcal{H})$ is a Gaussian dynamical semigroup if its infinitesimal generator \mathfrak{G} can be expressed in the form

$$\mathfrak{G} \hat{A} = -i[\hat{H}, \hat{A}] + \sum_{k=1}^{N^2-1} \gamma_k \left(\hat{F}_k \hat{A} \hat{F}_k - \frac{1}{2} (\hat{F}_k^2 \hat{A} + \hat{A} \hat{F}_k^2) \right), \tag{5.15}$$

where \hat{H} is a traceless self-adjoint operator, $\hat{F}_1, \dots, \hat{F}_{N^2-1}$ are traceless self-adjoint operators satisfying (5.14) and

$$\gamma_1 \geq 0, \dots, \gamma_{N^2-1} \geq 0, \quad \gamma_1 \gamma_2 \cdots \gamma_{N^2-1} \neq 0. \tag{5.16}$$

Otherwise stated, the infinitesimal generator \mathfrak{L} of formula (5.7) gives rise to a Gaussian dynamical semigroup if the completely positive map \mathfrak{F} admits a decomposition of the form (5.10) where the linear operators $\hat{F}_1, \dots, \hat{F}_{N^2-1}$ are—in addition to the previously mentioned assumptions—self-adjoint, and there is at least a nonzero number in the set $\{\gamma_1, \dots, \gamma_{N^2-1}\}$. Note that, according to proposition 5.2, every Gaussian dynamical semigroup is a random unitary semigroup. We will show, moreover, that every Gaussian dynamical semigroup arises in a natural way as a twirling semigroup associated with a convolution semigroup of measures of a certain type, namely, with a ‘Gaussian semigroup of measures’.

In order to define such a class of convolution semigroups of measures, let us consider the following set of probability measures on the Lie group G :

$$\mathcal{D}(G) := \{\delta_g : g \in G\} \subset \mathcal{M}^1(G), \tag{5.17}$$

i.e. $\mathcal{D}(G)$ is the set of all Dirac measures on G .

Definition 5.4. A continuous convolution semigroup of measures $\{\mu_t\}_{t \in \mathbb{R}^+}$ —such that, for $t > 0$, $\mu_t \in \mathcal{M}^1(G) \setminus \mathcal{D}(G)$ —is called a Gaussian (convolution) semigroup of measures if

$$\lim_{t \downarrow 0} t^{-1} \mu_t(\mathcal{U}\mathcal{E}_e) = 0 \tag{5.18}$$

for every Borel neighborhood of the identity \mathcal{E}_e in G .

The previous definition is originally due to Courrège [41] and Siebert [42]. Gaussian semigroups of measures on G describe the statistical properties of Brownian motion on G [5].

We have already encountered condition (5.18)—see (3.2)—in the case where $G = \mathbb{R}^n$. Thus, the reader should be familiar with its consequences. In general, it is a well-known fact—see [5]—that, given a Gaussian semigroup of measures $\{\mu_t\}_{t \in \mathbb{R}^+}$ on G , for every $t \in \mathbb{R}^+$ the measure μ_t has support contained in the connected component with the identity of G : $\text{supp}(\mu_t) \subset G_e$. Therefore, in the following we can assume without loss of generality that—as far as a Gaussian semigroup of measures is concerned—the group G is connected. It is a remarkable result—see, again, [5]—the representation kit $\{b^j, a^{jk}, \eta\}_{j,k=1}^n$ of a continuous convolution semigroup of measures on G corresponds to a Gaussian semigroup of measures if and only if

$$\eta = 0 \quad \text{and} \quad [a^{jk}]_{j,k=1}^n \neq 0. \tag{5.19}$$

This result implies, in particular, that Gaussian semigroups of measures do exist; precisely, one for each set $\{b^j, a^{jk}\}_{j,k=1}^n$, where $[a^{jk}]_{j,k=1}^n$ is a non-zero positive matrix. Note, moreover, that the Lévy–Kintchine formula (2.20) holds, in this case, with $\mathfrak{R} = 0$, i.e.

$$(\mathfrak{J}f)(g) = \sum_{j=1}^n b^j (\xi_j f)(g) + \sum_{j,k=1}^n a^{jk} (\xi_j \xi_k f)(g), \quad f \in \mathcal{C}_c^2(G). \tag{5.20}$$

Note, moreover, that Gaussian semigroups of measures on G form a special class among the convolution semigroups of measures of the *first kind* on G (see section 2).

At this point, in order to get to the main result of this section (theorem 5.1 below), we need to pass through four technical lemmas. We will denote by $\{\mu_t\}_{t \in \mathbb{R}^+}$ an *arbitrary* continuous convolution semigroup of measures on G , with representation kit $\{b^j, a^{jk}, \eta\}_{j,k=1}^n$, and by $\mathcal{L}(U, \{\mu_t\})$ the infinitesimal generator of the twirling semigroup associated with the pair $(U, \{\mu_t\}_{t \in \mathbb{R}^+})$.

Lemma 5.2. *Let $\varphi : G \rightarrow \mathbb{C}$ be a bounded Borel function, which vanishes on a Borel neighborhood of the identity of G . Then, for every sequence $\{\tau_m\}_{m \in \mathbb{N}}$ in \mathbb{R}_*^+ converging to zero, there is a subsequence $\{t_k \equiv \tau_{m_k}\}_{k \in \mathbb{N}}$ such that the limit*

$$\lim_{k \rightarrow \infty} \frac{1}{t_k} \int_G \varphi(g) \, d\mu_{t_k}(g) \tag{5.21}$$

exists in \mathbb{C} .

Proof. According to a well-known result—see [5], lemma 4.1.4—for every Borel neighborhood of the identity \mathcal{E}_e in G , we have

$$\sup_{t \in \mathbb{R}_*^+} t^{-1} \mu_t(\mathbb{C}\mathcal{E}_e) < \infty. \tag{5.22}$$

Thus, if $\varphi : G \rightarrow \mathbb{C}$ is a bounded Borel function vanishing on \mathcal{E}_e , we have

$$\sup_{t \in \mathbb{R}_*^+} t^{-1} \left| \int_G \varphi(g) \, d\mu_t(g) \right| \leq \sup_{t \in \mathbb{R}_*^+} t^{-1} \int_{\mathbb{C}\mathcal{E}_e} |\varphi(g)| \, d\mu_t(g) \leq \sup_{g \in G} |\varphi(g)| \sup_{t \in \mathbb{R}_*^+} t^{-1} \mu_t(\mathbb{C}\mathcal{E}_e) < \infty. \tag{5.23}$$

Now, take any sequence $\{\tau_m\}_{m \in \mathbb{N}}$ in \mathbb{R}_*^+ converging to zero. Relation (5.23) implies that

$$\sup_{m \in \mathbb{N}} \frac{1}{\tau_m} \left| \int_G \varphi(g) \, d\mu_{\tau_m}(g) \right| < \infty. \tag{5.24}$$

Then, by the Bolzano–Weierstrass theorem, there is a subsequence $\{t_k \equiv \tau_{m_k}\}_{k \in \mathbb{N}} \subset \mathbb{R}_*^+$ of $\{\tau_m\}_{m \in \mathbb{N}}$ such that the limit (5.21) exists in \mathbb{C} . The proof is complete. \square

The previous lemma will allow us to prove the following result, which will be fundamental for our purposes.

Lemma 5.3. *If $f : G \rightarrow \mathbb{C}$ is a bounded smooth function such that the limit*

$$\lim_{t \downarrow 0} \frac{1}{t} \left(\int_G f(g) d\mu_t(g) - f(e) \right) \tag{5.25}$$

exists in \mathbb{C} , then this limit is equal to

$$\sum_{j=1}^n b^j (\xi_j f)(e) + \sum_{j,k=1}^n a^{jk} (\xi_j \xi_k f)(e) + \int_{G_*} \left(f(g) - f(e) - \sum_{j=1}^n (\xi_j f)(e) \bar{x}^j(g) \right) d\eta(g). \tag{5.26}$$

Therefore, in the case where the convolution semigroup of measures $\{\mu_t\}_{t \in \mathbb{R}^+}$ is of the first kind (i.e. the associated Lévy measure η on G_ is finite), the limit (5.25)—if it exists—is given by*

$$\sum_{j=1}^n (b^j + c^j(\eta)) (\xi_j f)(e) + \sum_{j,k=1}^n a^{jk} (\xi_j \xi_k f)(e) + \int_{G_*} f(g) d\eta(g) - \eta(G_*) f(e), \tag{5.27}$$

where

$$c^j(\eta) := - \int_{G_*} \bar{x}^j(g) d\eta(g), \quad j = 1, \dots, n. \tag{5.28}$$

Proof. Since G (being locally compact and second countable) is σ -compact, there exists a sequence $\{\beta_m\}_{m \in \mathbb{N}}$ of non-negative smooth functions on G characterized as follows:

- (i) for every $m \in \mathbb{N}$, β_m belongs to $C_c^\infty(G; \mathbb{R})$ and $\beta_m(G) \subset [0, 1]$;
- (ii) there is a sequence $\{\mathcal{K}_m^\circ\}_{m \in \mathbb{N}}$ of precompact open subsets of G such that

$$e \in \mathcal{K}_1^\circ, \quad \mathcal{K}_1^\circ \subset \mathcal{K}_2^\circ \subset \dots, \quad \bigcup_{m=1}^\infty \mathcal{K}_m^\circ = G, \tag{5.29}$$

$$\beta_m(g) = 1, \quad \forall g \in \mathcal{K}_m, \tag{5.30}$$

where \mathcal{K}_m is the closure of the set \mathcal{K}_m° : $\mathcal{K}_m = \overline{\mathcal{K}_m^\circ}$; we can assume that

$$\mathcal{K}_1^\circ \supset \text{supp}(\bar{x}^1) \cup \dots \cup \text{supp}(\bar{x}^n); \tag{5.31}$$

- (iii) there is a sequence $\{\mathcal{O}_m\}_{m \in \mathbb{N}}$ of precompact open subsets of G such that for every $m \in \mathbb{N}$,

$$\mathcal{O}_m \supset \mathcal{K}_m \tag{5.32}$$

and

$$\beta_m(g) = 0, \quad \forall g \in \mathbb{C}\mathcal{O}_m. \tag{5.33}$$

In fact, as G is σ -compact, there exist sequences $\{\mathcal{K}_m^\circ\}_{m \in \mathbb{N}}$, $\{\mathcal{O}_m\}_{m \in \mathbb{N}}$ of precompact open subsets of G satisfying (5.29) and (5.32), respectively; relation (5.31) can always be satisfied by the compactness of the supports of the adapted coordinates. Next, by a standard procedure in the theory of smooth manifolds one constructs suitable ‘bump functions’ $\{\beta_m\}_{m \in \mathbb{N}}$, contained in $C_c^\infty(G; \mathbb{R})$, satisfying (5.30) and (5.33).

By the existence of the limit (5.25), applying lemma 5.2 to the bounded smooth function $f(1 - \beta_1)$ (which vanishes on the compact neighborhood \mathcal{K}_1 of e), for some sequence $\{t_k\}_{k \in \mathbb{N}}$ in \mathbb{R}_*^+ converging to zero we have

$$\begin{aligned} \lim_{t \downarrow 0} \frac{1}{t} \left(\int_G f(g) d\mu_t(g) - f(e) \right) &= \lim_{k \rightarrow \infty} \frac{1}{t_k} \left(\int_G f(g) d\mu_{t_k}(g) - f(e) \right) \\ &= \lim_{k \rightarrow \infty} \frac{1}{t_k} \left(\int_G f(g) \beta_1(g) d\mu_{t_k}(g) - f(e) \right) \\ &\quad + \lim_{k \rightarrow \infty} \frac{1}{t_k} \int_G f(g) (1 - \beta_1(g)) d\mu_{t_k}(g), \end{aligned} \tag{5.34}$$

where, since the function $f\beta_1$ belongs to $C_c^\infty(G; \mathbb{C})$ and $\beta_1(e) = 1$, the first limit in the last member of (5.34) exists and is equal to $(\mathfrak{J}(f\beta_1))(e)$, with \mathfrak{J} denoting the generator of the probability semigroup associated with $\{\mu_t\}_{t \in \mathbb{R}_*^+}$. We stress that the sequence $\{t_k\}_{k \in \mathbb{N}}$ can be extracted, as a subsequence, from any sequence of strictly positive numbers converging to zero. Thus, we find that

$$\lim_{t \downarrow 0} \frac{1}{t} \left(\int_G f(g) d\mu_t(g) - f(e) \right) = (\mathfrak{J}(f\beta_1))(e) + \lim_{k \rightarrow \infty} \frac{1}{t_k} \int_G f(g) (1 - \beta_1(g)) d\mu_{t_k}(g), \tag{5.35}$$

where, by virtue of the Lévy–Kintchine formula applied to the function $f\beta_1 \in C_c^\infty(G; \mathbb{C})$ (note that $(f\beta_1)(g) = f(g)$, for $g \in \mathcal{K}_1^\circ$), we can write

$$\begin{aligned} (\mathfrak{J}(f\beta_1))(e) &= \sum_{j=1}^n b^j (\xi_j f)(e) + \sum_{j,k=1}^n a^{jk} (\xi_j \xi_k f)(e) \\ &\quad + \int_{G_*} \left((f\beta_1)(g) - f(e) - \sum_{j=1}^n (\xi_j f)(e) \bar{x}^j(g) \right) d\eta(g). \end{aligned} \tag{5.36}$$

At this point, in order to evaluate the last term in (5.36), it will be convenient to set

$$\begin{aligned} \varphi_{1,1}(g) &\equiv f(g)\beta_1(g), \quad \text{and, for } m \in \mathbb{N}, \quad m \geq 2, \\ \varphi_{1,1}(g) &= \varphi_{m,1}(g) + \varphi_{m,2}(g), \quad \varphi_{m,1}(g) := f(g)\beta_m(g), \\ \varphi_{m,2}(g) &:= f(g) (\beta_1(g) - \beta_m(g)). \end{aligned} \tag{5.37}$$

Clearly, the functions $\{\varphi_{m,1}\}_{m \geq 1}$ belong to $C_c^\infty(G; \mathbb{C})$. It is easy to check that the functions $\{\varphi_{m,2}\}_{m \geq 2}$ belong to $C_c^\infty(G; \mathbb{C})$, as well. Indeed, they are obviously smooth and

$$\text{supp}(\beta_1 - \beta_m) \subset \overline{\mathcal{K}_1 \cap (\mathcal{O}_1 \cup \mathcal{O}_m)} \subset \overline{\mathcal{K}_1} \cap \overline{(\mathcal{O}_1 \cup \mathcal{O}_m)} = \overline{\mathcal{K}_1} \cap (\overline{\mathcal{O}_1} \cup \overline{\mathcal{O}_m}). \tag{5.38}$$

Thus, the set $\text{supp}(\beta_1 - \beta_m)$ is compact in G . Note that, as it does not contain the identity, it is a compact set in G_* , as well; hence $\{\varphi_{m,2}\}_{m \geq 2} \subset C_c^\infty(G_*; \mathbb{C})$. This fact allows us to use formula (2.22) in such a way to decompose the last term in (5.36) as follows:

$$\begin{aligned} &\int_{G_*} \left(\varphi_{1,1}(g) - f(e) - \sum_{j=1}^n (\xi_j f)(e) \bar{x}^j(g) \right) d\eta(g) \\ &= \int_{G_*} \left(\varphi_{m,1}(g) - f(e) - \sum_{j=1}^n (\xi_j f)(e) \bar{x}^j(g) \right) d\eta(g) \\ &\quad + \lim_{k \rightarrow \infty} \frac{1}{t_k} \int_G \varphi_{m,2}(g) d\mu_{t_k}(g) \equiv \varkappa, \quad m \geq 2. \end{aligned} \tag{5.39}$$

Note that the number κ does not depend on the index m . At this point, considering the last term in (5.35), for every $m \geq 2$ we have

$$\begin{aligned} \kappa + \lim_{k \rightarrow \infty} \frac{1}{t_k} \int_G f(g) (1 - \beta_1(g)) d\mu_{t_k}(g) &= \int_{G_*} \left(\varphi_{m,1}(g) - f(e) - \sum_{j=1}^n (\xi_j f)(e) \bar{x}^j(g) \right) d\eta(g) \\ &+ \lim_{k \rightarrow \infty} \frac{1}{t_k} \int_G f(g) (1 - \beta_m(g)) d\mu_{t_k}(g). \end{aligned} \tag{5.40}$$

The rhs of relation (5.40) can be regarded as the (constant) sum of two sequences labeled by the index m . Therefore, if one of the two sequences is converging, the other one must converge too. Let us prove that the limit

$$\lim_{m \rightarrow \infty} \int_{G_*} \left(\varphi_{m,1}(g) - f(e) - \sum_{j=1}^n (\xi_j f)(e) \bar{x}^j(g) \right) d\eta(g) \tag{5.41}$$

exists and is equal to

$$\int_{G_*} \left(f(g) - f(e) - \sum_{j=1}^n (\xi_j f)(e) \bar{x}^j(g) \right) d\eta(g). \tag{5.42}$$

Indeed—observing that, by (5.31), $\bar{x}^j(g) = \bar{x}^j(g)\beta_m(g)$, and denoting by $\chi_{\mathcal{C}\mathcal{K}_m}$ the characteristic function of the set $\mathcal{C}\mathcal{K}_m$ —we can write the estimate

$$\begin{aligned} &\left| \varphi_{m,1}(g) - f(e) - \sum_{j=1}^n (\xi_j f)(e) \bar{x}^j(g) \right| \\ &= \left| f(g)\beta_m(g) - f(e) - \sum_{j=1}^n (\xi_j f)(e) \bar{x}^j(g)\beta_m(g) \right| \\ &\leq \left| f(g) - f(e) - \sum_{j=1}^n (\xi_j f)(e) \bar{x}^j(g) \right| \beta_m(g) + |f(e)|(1 - \beta_m(g)) \\ &\leq \left| f(g) - f(e) - \sum_{j=1}^n (\xi_j f)(e) \bar{x}^j(g) \right| + |f(e)|\chi_{\mathcal{C}\mathcal{K}_m}(g) \end{aligned} \tag{5.43}$$

for all $m \in \mathbb{N}$ and $g \in G$. Therefore, since $\chi_{\mathcal{C}\mathcal{K}_m} \leq \chi_{\mathcal{C}\mathcal{K}_1}$, we find out that

$$\begin{aligned} &\left| \varphi_{m,1}(g) - f(e) - \sum_{j=1}^n (\xi_j f)(e) \bar{x}^j(g) \right| \\ &\leq \left| f(g) - f(e) - \sum_{j=1}^n (\xi_j f)(e) \bar{x}^j(g) \right| + |f(e)|\chi_{\mathcal{C}\mathcal{K}_1}(g). \end{aligned} \tag{5.44}$$

The expression on the rhs of (5.44) defines a function contained in $L^1(G_*, \eta; \mathbb{C})$. Therefore, since $\lim_{m \rightarrow \infty} \beta_m(g) = 1$, for all $g \in G$, by the ‘dominated convergence theorem’ the limit (5.41) exists and is equal to (5.42), as claimed.

Let us resume what we have obtained up to this point. By relations (5.35), (5.36), (5.39) and (5.40), and by the fact that the limit (5.41) is equal to (5.42), we conclude that the existence of the limit (5.25), for a bounded smooth function $f : G \rightarrow \mathbb{C}$, implies that this limit must coincide with

$$\begin{aligned} & \sum_{j=1}^n b^j (\xi_j f)(e) + \sum_{j,k=1}^n a^{jk} (\xi_j \xi_k f)(e) + \int_{G_*} \left(f(g) - f(e) - \sum_{j=1}^n (\xi_j f)(e) \bar{x}^j(g) \right) d\eta(g) \\ & + \lim_{m \rightarrow \infty} \lim_{k \rightarrow \infty} \frac{1}{t_k} \int_G f(g) (1 - \beta_m(g)) d\mu_{t_k}(g), \end{aligned} \tag{5.45}$$

for some sequence $\{t_k\}_{k \in \mathbb{N}}$ in \mathbb{R}_*^+ converging to zero that can be extracted, as a subsequence, from any sequence of strictly positive numbers converging to zero. Note that the iterated limit above *must* exist (as the first member of (5.40) does not depend on m and the limit (5.41) exists).

We now apply this result to the function $f \equiv 1$. Then, we find immediately that

$$\lim_{m \rightarrow \infty} \lim_{k \rightarrow \infty} \frac{1}{t_k} \int_G (1 - \beta_m(g)) d\mu_{t_k}(g) = 0, \tag{5.46}$$

for some sequence $\{t_k\}_{k \in \mathbb{N}}$ in \mathbb{R}_*^+ converging to zero.

Finally, considering again an *arbitrary* bounded smooth function f on G for which the limit (5.25) exists, extract from $\{t_k\}_{k \in \mathbb{N}}$ a subsequence $\{t_k\}_{k \in \mathbb{N}}$ such that this limit coincides with (5.45). From (5.46)—observing that the inequality

$$\left| \int_G f(g) (1 - \beta_m(g)) d\mu_{t_k}(g) \right| \leq \|f\|_{\text{sup}} \int_G (1 - \beta_m(g)) d\mu_{t_k}(g) \tag{5.47}$$

implies

$$\begin{aligned} & \left| \lim_{m \rightarrow \infty} \lim_{k \rightarrow \infty} \frac{1}{t_k} \int_G f(g) (1 - \beta_m(g)) d\mu_{t_k}(g) \right| = \lim_{m \rightarrow \infty} \lim_{k \rightarrow \infty} \frac{1}{t_k} \left| \int_G f(g) (1 - \beta_m(g)) d\mu_{t_k}(g) \right| \\ & \leq \|f\|_{\text{sup}} \lim_{m \rightarrow \infty} \lim_{k \rightarrow \infty} \frac{1}{t_k} \int_G (1 - \beta_m(g)) d\mu_{t_k}(g) \end{aligned} \tag{5.48}$$

—we conclude that the last term in (5.45) vanishes and the proof is complete. □

The next lemma will lead us very close to the main result of this section.

Lemma 5.4. *With the previous notations and assumptions, for every operator $\hat{A} \in \mathcal{B}(\mathcal{H})$, the following relation holds:*

$$\begin{aligned} \mathfrak{L}(U, \{\mu_t\})\hat{A} &= \lim_{t \downarrow 0} \frac{1}{t} \left(\int_G d\mu_t(g) U(g)\hat{A}U(g)^* - \hat{A} \right) \\ &= \sum_{j=1}^n b^j [\hat{X}_j, \hat{A}] + \sum_{j,k=1}^n a^{jk} (\{\hat{X}_j \hat{X}_k, \hat{A}\} - 2\hat{X}_j \hat{A} \hat{X}_k) \\ &+ \int_{G_*} \left(U(g)\hat{A}U(g)^* - \hat{A} - \sum_{j=1}^n \bar{x}^j(g) [\hat{X}_j, \hat{A}] \right) d\eta(g) \equiv \hat{A}', \end{aligned} \tag{5.49}$$

where $\{\cdot, \cdot\}$ is the anti-commutator and the set $\{\hat{X}_1, \dots, \hat{X}_n\} \subset \mathfrak{i}\mathcal{B}_{\mathbb{R}}(\mathcal{H})$ is the n -tuple of operators defined by (2.16). Suppose, in particular, that $\{\mu_t\}_{t \in \mathbb{R}^+}$ is a convolution semigroup of measures of the first kind. Then, for every $\hat{A} \in \mathcal{B}(\mathcal{H})$, we have

$$\begin{aligned} \lim_{t \downarrow 0} \frac{1}{t} \left(\int_G d\mu_t(g) U(g)\hat{A}U(g)^* - \hat{A} \right) &= \sum_{j=1}^n (b^j + c^j(\eta)) [\hat{X}_j, \hat{A}] \\ &+ \sum_{j,k=1}^n a^{jk} (\{\hat{X}_j \hat{X}_k, \hat{A}\} - 2\hat{X}_j \hat{A} \hat{X}_k) + \eta(G_*) (\mathfrak{U}_\eta^U - I) \hat{A}, \end{aligned} \tag{5.50}$$

where $\{c^j(\eta)\}_{j=1}^n$ are the real numbers defined by (5.28), and $\mathfrak{L}_\eta^U : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ is identically zero for $\eta = 0$ and a random unitary map for $\eta \neq 0$, with

$$\mathfrak{L}_\eta^U = \eta(G_*)^{-1} \int_{G_*} d\eta(g) U \vee U(g), \quad \eta \neq 0. \tag{5.51}$$

Proof. It is sufficient to show that

$$\begin{aligned} \langle \phi, (\mathfrak{L}(U, \{\mu_t\})\hat{A})\psi \rangle &= \lim_{t \downarrow 0} \frac{1}{t} \left(\int_G d\mu_t(g) \langle \phi, U(g)\hat{A}U(g)^*\psi \rangle - \langle \phi, \hat{A}\psi \rangle \right) \\ &= \langle \phi, \hat{A}'\psi \rangle \end{aligned} \tag{5.52}$$

for arbitrary $\hat{A} \in \mathcal{B}(\mathcal{H})$ and $\phi, \psi \in \mathcal{H}$, where \hat{A}' is the shorthand notation introduced in (5.49). To this aim, since the limit in (5.52) exists, we can apply lemma 5.3 to the bounded smooth function $f : G \rightarrow \mathbb{C}$ defined by

$$f(g) := \langle \phi, U(g)\hat{A}U(g)^*\psi \rangle. \tag{5.53}$$

Using the notation introduced in section 2, there exists a neighborhood of the identity \mathcal{E}_e in G such that

$$U(g) = e^{\bar{x}^1(g)\hat{X}_1 + \dots + \bar{x}^n(g)\hat{X}_n}, \quad \forall g \in \mathcal{E}_e. \tag{5.54}$$

Therefore, we have that

$$\xi_j \langle \phi, U(g)\hat{A}U(g)^*\psi \rangle|_{g=e} = \langle \phi, [\hat{X}_j, \hat{A}]\psi \rangle, \tag{5.55}$$

$$\xi_j \xi_k \langle \phi, U(g)\hat{A}U(g)^*\psi \rangle|_{g=e} = \langle \phi, (\hat{X}_j \hat{X}_k \hat{A} + \hat{A} \hat{X}_k \hat{X}_j - \hat{X}_j \hat{A} \hat{X}_k - \hat{X}_k \hat{A} \hat{X}_j)\psi \rangle. \tag{5.56}$$

Now, exploiting formula (5.26) and the fact that the matrix $[a^{jk}]_{j,k=1}^n$ is symmetric, we obtain immediately relation (5.52). \square

The last technical lemma will establish a useful link between the generator of the twirling semigroup associated with the pair $(U, \{\mu_t\}_{t \in \mathbb{R}^+})$ —with $\{\mu_t\}_{t \in \mathbb{R}^+}$ denoting a *generic* continuous convolution semigroup of measures on G —and the convolution semigroups of measures on G of the first kind.

Lemma 5.5. *There exists a sequence $\{\{\mu_{t;m}\}_{t \in \mathbb{R}^+} : m \in \mathbb{N}\}$ of continuous convolution semigroups of measures of the first kind on G —with $\{\mu_{t;m}\}_{t \in \mathbb{R}^+}$ having a representation kit of the form $\{b^j, a^{jk}, \eta_m\}_{j,k=1}^n$ —such that*

$$\lim_{m \rightarrow \infty} \mathfrak{L}(U, \{\mu_{t;m}\}) = \mathfrak{L}(U, \{\mu_t\}), \quad \text{and} \quad \lim_{m \rightarrow \infty} \int_{G_*} f(g) d\eta_m(g) = \int_{G_*} f(g) d\eta(g), \tag{5.57}$$

for every bounded Borel function $f : G_* \rightarrow \mathbb{C}$ belonging to $L^1(G_*, \eta; \mathbb{C})$.

Proof. Let $\{b^j, a^{jk}, \eta\}_{j,k=1}^n$ denote, as usual, the representation kit of the convolution semigroup of measures $\{\mu_t\}_{t \in \mathbb{R}^+}$, and let $\Phi : G \rightarrow \mathbb{R}^+$ be a Hunt function and Φ' its restriction to G_* . For every $m \in \mathbb{N}$, consider the measure η_m on G_* determined by

$$d\eta_m(g) = (1 - \exp(-m\Phi'(g))) d\eta(g), \quad g \in G_*. \tag{5.58}$$

The measure η_m is finite (by construction), for all $m \in \mathbb{N}$, and, as $0 \leq (1 - \exp(-m\Phi'(g))) \leq 1$, by the ‘dominated convergence theorem’ we have that

$$\lim_{m \rightarrow \infty} \int_{G_*} f(g) d\eta_m(g) = \int_{G_*} f(g) d\eta(g), \tag{5.59}$$

for every bounded Borel function $f : G_* \rightarrow \mathbb{C}$ contained in $L^1(G_*, \eta; \mathbb{C})$. Denote by $\{\mu_{t;m}\}_{t \in \mathbb{R}^+}$ the continuous convolution semigroup of measures with representation kit $\{b^j, a^{jk}, \eta_m\}_{j,k=1}^n$. From relations (5.49) and (5.59)—setting $f(g) = \langle \phi, U(g)\hat{A}U(g)^* - \hat{A} - \sum_{j=1}^n \bar{x}^j(g)[\hat{X}_j, \hat{A}]\psi \rangle$, $g \in G_*$, for any $\hat{A} \in \mathcal{B}(\mathcal{H})$ and $\phi, \psi \in \mathcal{H}$ —we obtain that

$$\lim_{m \rightarrow \infty} \mathcal{L}(U, \{\mu_{t;m}\}) = \mathcal{L}(U, \{\mu_t\}). \tag{5.60}$$

The proof is complete. □

Having completed the main technical proofs, we are finally ready to focus on the main result of this section, which can be regarded as a generalization of an already cited classical result of Kümmerer and Maassen [17]. The latter result is obtained from the former (namely theorem 5.1 below) by choosing the unitary representation U as the defining representation of $SU(N)$ (up to unitary equivalence). It will be now convenient to establish a few additional notations. Given a nonempty subset \mathcal{S} of $\mathcal{U}(\mathcal{H})$, we will denote by $\text{cone}(\mathcal{S})$ the cone in $\mathcal{L}(\mathcal{H})$ generated by this set—i.e. $\text{cone}(\mathcal{S}) := \mathbb{R}^+ \mathcal{S}$ —and by $\overline{\text{cone}}(\mathcal{S})$ the closure of such cone. If $0 \in \mathcal{S}$, consider, moreover, the set

$$\text{cone}_0(\mathcal{S}) := \{\mathfrak{A} \in \mathcal{L}(\mathcal{H}) : \exists \{\alpha_m\}_{m \in \mathbb{N}} \subset \mathbb{R}_*^+, \alpha_m \rightarrow \infty, \exists \{\mathfrak{A}_m\}_{m \in \mathbb{N}} \subset \mathcal{S} \text{ s.t. } \alpha_m \mathfrak{A}_m \rightarrow \mathfrak{A}\}. \tag{5.61}$$

It can be shown that if \mathcal{S} is a closed set, then $\text{cone}_0(\mathcal{S})$ is a closed cone (see [43], where a closed subset of a normed vector space is considered). Denoting, as above, by \mathcal{V} a subgroup of $\mathcal{U}(\mathcal{H})$ and by $\overline{\mathcal{V}}$ the subgroup of $\mathcal{U}(\mathcal{H})$ which is the closure of \mathcal{V} , the sets $\overline{\text{cone}}(\overline{\mathcal{V}} \vee \overline{\mathcal{V}} - I)$ and $\mathcal{C}(\mathcal{V}) := \overline{\text{co-cone}}(\mathcal{V} \vee \mathcal{V} - I)$ are characterized as follows.

Proposition 5.3. *For the closed convex cone $\mathcal{C}(\mathcal{V}) := \overline{\text{co-cone}}(\mathcal{V} \vee \mathcal{V} - I)$ we have*

$$\mathcal{C}(\mathcal{V}) = \overline{\text{co-cone}}(\overline{\mathcal{V}} \vee \overline{\mathcal{V}} - I) = \overline{\text{co-cone}}(\text{DM}_{\text{ru}}(\mathcal{V}) - I) = \overline{\text{co-cone}}(\text{DM}_{\text{ru}}(\overline{\mathcal{V}}) - I). \tag{5.62}$$

The set $\text{cone}_0(\overline{\mathcal{V}} \vee \overline{\mathcal{V}} - I)$ is a closed cone in $\mathcal{L}(\mathcal{H})$. The closed cone $\overline{\text{cone}}(\overline{\mathcal{V}} \vee \overline{\mathcal{V}} - I)$ is contained in $\mathcal{C}(\mathcal{V})$ and

$$\overline{\text{cone}}(\overline{\mathcal{V}} \vee \overline{\mathcal{V}} - I) = \text{cone}_0(\overline{\mathcal{V}} \vee \overline{\mathcal{V}} - I) \cup \text{cone}(\overline{\mathcal{V}} \vee \overline{\mathcal{V}} - I). \tag{5.63}$$

Proof. The proof of relations (5.62) goes as follows. First observe that

$$\mathcal{C}(\mathcal{V}) := \overline{\text{co-cone}}(\mathcal{V} \vee \mathcal{V} - I) = \overline{\text{co-cone}}(\overline{\mathcal{V}} \vee \overline{\mathcal{V}} - I) = \overline{\text{co-cone}}(\overline{\mathcal{V}} \vee \overline{\mathcal{V}} - I). \tag{5.64}$$

Next, we have

$$\begin{aligned} \overline{\text{co-cone}}(\mathcal{V} \vee \mathcal{V} - I) &= \overline{\text{co-cone}}(\text{co}(\mathcal{V} \vee \mathcal{V} - I)) \\ &= \overline{\text{co-cone}}(\text{DM}_{\text{ru}}(\mathcal{V}) - I) = \overline{\text{co-cone}}(\overline{\text{DM}_{\text{ru}}(\mathcal{V}) - I}) = \overline{\text{co-cone}}(\text{DM}_{\text{ru}}(\overline{\mathcal{V}}) - I). \end{aligned} \tag{5.65}$$

Thus, the proof of (5.62) is complete.

Next, since $\overline{\mathcal{V}} \vee \overline{\mathcal{V}} - I$ is a closed set, $\text{cone}_0(\overline{\mathcal{V}} \vee \overline{\mathcal{V}} - I)$ is a closed cone, and from our previous arguments it is clear that the closed cone $\overline{\text{cone}}(\overline{\mathcal{V}} \vee \overline{\mathcal{V}} - I)$ is contained in $\mathcal{C}(\mathcal{V})$. Let us prove relation (5.63). For every compact subset \mathcal{K} of $\mathcal{L}(\mathcal{H})$ such that $0 \in \mathcal{K}$, the following decomposition holds: $\overline{\text{cone}}(\mathcal{K}) = \text{cone}_0(\mathcal{K}) + \text{cone}(\mathcal{K})$ (see [43], theorem 3.2, and take into account the fact that the ‘asymptotic cone’—or ‘recession cone’—generated by a bounded set coincides with the origin). Apply this result to the compact set $\overline{\mathcal{V}} \vee \overline{\mathcal{V}} - I$. The proof is complete. □

In the following, the subgroups \mathcal{V} and $\overline{\mathcal{V}}$ of $\mathcal{U}(\mathcal{H})$ will be identified with the subgroups $U(G)$ and $\overline{U(G)}$, respectively. Let V_U be the real vector space obtained by projecting

$i(\text{Ran}(\pi_U))$ —regarded as a vector subspace of $\mathcal{B}_{\mathbb{R}}(\mathcal{H})$ —onto the orthogonal complement of the one-dimensional space spanned by the identity, namely

$$V_U := \{\hat{A} \in \mathcal{B}_{\mathbb{R}}(\mathcal{H}) : \hat{A} = i(\pi_U(\xi) - N^{-1} \text{tr}(\pi_U(\xi))I), \xi \in \text{Lie}(G)\}. \quad (5.66)$$

We will denote by D the dimension of the vector space V_U ($D \leq \min\{n, N - 1\}$). Observe that, if G is a semisimple Lie group, then $[\text{Lie}(G), \text{Lie}(G)] = \text{Lie}(G)$ and $V_U = i(\text{Ran}(\pi_U))$. Finally, in the case where $\{\mu_t\}_{t \in \mathbb{R}^+}$ is of regular type, the adapted coordinates $\{g \mapsto \bar{x}^1(g), \dots, g \mapsto \bar{x}^n(g)\}$ are integrable with respect to the Lévy measure η and we can set

$$c^j(\eta) := - \int_{G_*} \bar{x}^j(g) d\eta(g), \quad j = 1, \dots, n. \quad (5.67)$$

Theorem 5.1. *Let G be a Lie group and U a smooth unitary representation of G in the Hilbert space \mathcal{H} . Then, for every continuous convolution semigroup of measures $\{\mu_t\}_{t \in \mathbb{R}^+}$ on G —let $\{b^j, a^{jk}, \eta\}_{j,k=1}^n$ be the associated representation kit—the infinitesimal generator $\mathfrak{L}(U, \{\mu_t\}) : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ of the twirling semigroup $\{\mathfrak{S}_t\}_{t \in \mathbb{R}^+}$ associated with the pair $(\{\mu_t\}_{t \in \mathbb{R}^+}, U)$ is of the form*

$$\mathfrak{L}(U, \{\mu_t\}) = \mathfrak{G}(U, \{\mu_t\}) + \mathfrak{W}(U, \{\mu_t\}), \quad (5.68)$$

where $\mathfrak{G}(U, \{\mu_t\})$ and $\mathfrak{W}(U, \{\mu_t\})$ belong to the closed convex cone $\mathcal{C}(U(G)) \subset \mathcal{C}(\mathcal{H}) \subset \mathcal{L}(\mathcal{H})$ and are given by

$$\mathfrak{G}(U, \{\mu_t\}) := \sum_{j=1}^n b^j [\hat{X}_j, (\cdot)] + \sum_{j,k=1}^n a^{jk} (\{\hat{X}_j \hat{X}_k, (\cdot)\} - 2\hat{X}_j(\cdot) \hat{X}_k), \quad (5.69)$$

$$\mathfrak{W}(U, \{\mu_t\}) := \int_{G_*} \left(U \vee U(g) - I - \sum_{j=1}^n \bar{x}^j(g) [\hat{X}_j, (\cdot)] \right) d\eta(g), \quad (5.70)$$

with $\hat{X}_1, \dots, \hat{X}_n$ the skew-adjoint operators defined by (2.16). In the case where the semigroup of measures $\{\mu_t\}_{t \in \mathbb{R}^+}$ is of the first kind, we have

$$\mathfrak{W}(U, \{\mu_t\}) = \eta(G_*) (\mathfrak{U}_\eta^U - I) + \sum_{j=1}^n c^j(\eta) [\hat{X}_j, (\cdot)], \quad (5.71)$$

with $\mathfrak{U}_\eta^U : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ identically zero, for $\eta = 0$, and

$$\mathfrak{U}_\eta^U := \eta(G_*)^{-1} \int_{G_*} U \vee U(g) d\eta(g) \in \text{DM}_{\text{ru}}(\overline{U(G)}), \quad \text{for } \eta \neq 0. \quad (5.72)$$

Suppose, instead, that the semigroup of measures $\{\mu_t\}_{t \in \mathbb{R}^+}$ is of the second kind. Then, there exists a sequence $\{\{\mu_{t,m}\}_{t \in \mathbb{R}^+} : m \in \mathbb{N}\}$ of continuous convolution semigroups of measures of the first kind on G —with $\{\mu_{t,m}\}_{t \in \mathbb{R}^+}$ having a representation kit of the form $\{b^j, a^{jk}, \eta_m\}_{j,k=1}^n$ —such that $\lim_{m \rightarrow \infty} \mathfrak{L}(U, \{\mu_{t,m}\}) = \mathfrak{L}(U, \{\mu_t\})$, and $\lim_{m \rightarrow \infty} \int_{G_*} f(g) d\eta_m(g) = \int_{G_*} f(g) d\eta(g)$, for every bounded Borel function $f : G_* \rightarrow \mathbb{C}$ belonging to $L^1(G_*, \eta; \mathbb{C})$. Moreover, we have that

$$\mathfrak{W}(U, \{\mu_t\}) = \lim_{m \rightarrow \infty} \left(\eta_m(G_*) (\mathfrak{U}_{\eta_m}^U - I) + \sum_{j=1}^n c^j(\eta_m) [\hat{X}_j, (\cdot)] \right), \quad (5.73)$$

and, in the case where $\{\mu_t\}_{t \in \mathbb{R}^+}$ is of regular type,

$$\mathfrak{W}(U, \{\mu_t\}) = \mathfrak{W}_0(U, \{\mu_t\}) + \sum_{j=1}^n c^j(\eta) [\hat{X}_j, (\cdot)], \quad (5.74)$$

with $\mathfrak{W}_0(U, \{\mu_t\})$ denoting the element of the closed convex cone $\mathcal{C}(U(G))$ determined by

$$\mathfrak{W}_0(U, \{\mu_t\}) = \lim_{m \rightarrow \infty} \eta_m(G_*) (\mathfrak{U}_{\eta_m}^U - I). \tag{5.75}$$

The superoperator defined by (5.69) can be expressed in the canonical form

$$\mathfrak{G}(U, \{\mu_t\}) = -i[\hat{H}, (\cdot)] + \sum_{k=1}^D \gamma_k \left(\hat{F}_k(\cdot) \hat{F}_k - \frac{1}{2} (\hat{F}_k^2(\cdot) + (\cdot) \hat{F}_k^2) \right), \quad \gamma_k \geq 0, \tag{5.76}$$

with $\hat{H}, \hat{F}_1, \dots, \hat{F}_D$ traceless self-adjoint operators in \mathcal{H} satisfying

$$\hat{H}, \hat{F}_1, \dots, \hat{F}_D \subset \mathfrak{V}_U, \quad \langle \hat{F}_j, \hat{F}_k \rangle_{\text{HS}} = \delta_{jk}, \quad j, k = 1, \dots, D. \tag{5.77}$$

In particular, if $\{\mu_t\}_{t \in \mathbb{R}^+}$ is a Gaussian semigroup of measures, then $\mathfrak{W}(U, \{\mu_t\}) = 0$ and $\mathfrak{G}(U, \{\mu_t\})$ is of the form (5.15), i.e. $\{\mathfrak{S}_t\}_{t \in \mathbb{R}^+}$ is a Gaussian dynamical semigroup. Finally, for every superoperator $\mathfrak{L} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ of the form $\mathfrak{L} = \mathfrak{G} + \gamma_0(\mathfrak{U} - I)$ —with \mathfrak{G} of the general form given by the rhs of (5.76), \mathfrak{U} belonging to $\text{DM}_{\text{ru}}(U(G))$ and $\gamma_0 \geq 0$ —there is a continuous convolution semigroup of measures $\{\mu_t\}_{t \in \mathbb{R}^+}$ on G —with associated Lévy measure identically zero, if $\gamma_0(\mathfrak{U} - I) = 0$ —such that the infinitesimal generator of the twirling semigroup $\{\mathfrak{S}_t\}_{t \in \mathbb{R}^+}$ induced by the pair $(\{\mu_t\}_{t \in \mathbb{R}^+}, U)$ is \mathfrak{L} .

Proof. By lemma 5.4, the infinitesimal generator $\mathfrak{L}(U, \{\mu_t\})$ is of the form (5.68). In particular, in the case where the semigroup of measures $\{\mu_t\}_{t \in \mathbb{R}^+}$ is of the first kind, the superoperator $\mathfrak{W}(U, \{\mu_t\})$ is of the form (5.71). By lemma 5.5, in the case where the semigroup of measures $\{\mu_t\}_{t \in \mathbb{R}^+}$ is of the second kind, there exists a sequence $\{\{\mu_{t,m}\}_{t \in \mathbb{R}^+} : m \in \mathbb{N}\}$ of continuous convolution semigroups of measures of the first kind on G —with $\{\mu_{t,m}\}_{t \in \mathbb{R}^+}$ having a representation kit of the form $\{b^j, a^{jk}, \eta_m\}_{j,k=1}^n$ —such that

$$\lim_{m \rightarrow \infty} \mathfrak{L}(U, \{\mu_{t,m}\}) = \mathfrak{L}(U, \{\mu_t\}), \quad \text{and} \quad \lim_{m \rightarrow \infty} \int_{G_*} f(g) d\eta_m(g) = \int_{G_*} f(g) d\eta(g), \tag{5.78}$$

for every bounded Borel function $f : G_* \rightarrow \mathbb{C}$ belonging to $L^1(G_*, \eta; \mathbb{C})$. It follows that (5.73)—and, in the case where $\{\mu_t\}_{t \in \mathbb{R}^+}$ is of regular type, as $\lim_{m \rightarrow \infty} c^j(\eta_m) = c^j(\eta)$, (5.74)—hold true.

Let us prove that the superoperators $\mathfrak{G}(U, \{\mu_t\})$ and $\mathfrak{W}(U, \{\mu_t\})$ of decomposition (5.68) belong to the convex cone $\mathcal{C}(U(G))$. Indeed, diagonalizing the positive matrix $[a^{jk}]_{j,k=1}^n$ and introducing a suitable new basis $\{v_1, \dots, v_n\}$ in $\text{Lie}(G)$, we can write $\mathfrak{G}(U, \{\mu_t\})$ in the form

$$\mathfrak{G}(U, \{\mu_t\}) := [\hat{Y}_0, (\cdot)] + \sum_{j=1}^n \lambda_j (\{\hat{Y}_j \hat{Y}_j, (\cdot)\} - 2\hat{Y}_j(\cdot) \hat{Y}_j), \quad \lambda_j \geq 0, \tag{5.79}$$

where $\hat{Y}_0 \in \text{Ran}(\pi_U)$, $\hat{Y}_0 = \pi_U(v_0)$ (for some $v_0 \in \text{Lie}(G)$), and $\hat{Y}_1 = \pi_U(v_1), \dots, \hat{Y}_n = \pi_U(v_n)$ are skew-adjoint operators in \mathcal{H} . For the superoperator $[\hat{Y}_0, (\cdot)]$ we have

$$[\hat{Y}_0, (\cdot)] = \frac{d}{dt} (e^{t\hat{Y}_0}(\cdot) e^{-t\hat{Y}_0})|_{t=0} = \lim_{t \downarrow 0} t^{-1} (e^{t\hat{Y}_0}(\cdot) e^{-t\hat{Y}_0} - (\cdot)), \quad e^{\pm t\hat{Y}_0} = U(\exp_G(\pm t v_0)). \tag{5.80}$$

Therefore, $[\hat{Y}_0, (\cdot)]$ belongs to $\mathcal{C}(U(G))$. Analogously, since $e^{\pm t\hat{Y}_j} = U(\exp_G(\pm t v_j))$, we have that

$$\begin{aligned} \{\hat{Y}_j \hat{Y}_j, (\cdot)\} - 2\hat{Y}_j(\cdot) \hat{Y}_j &= \frac{1}{2} [\hat{Y}_j, [\hat{Y}_j, (\cdot)]] \\ &= \frac{1}{2} \frac{d^2}{dt^2} (e^{t\hat{Y}_j}(\cdot) e^{-t\hat{Y}_j}) \Big|_{t=0} \\ &= \lim_{t \downarrow 0} \frac{1}{2t^2} ((e^{t\hat{Y}_j}(\cdot) e^{-t\hat{Y}_j} - (\cdot)) + (e^{-t\hat{Y}_j}(\cdot) e^{t\hat{Y}_j} - (\cdot))) \in \mathcal{C}(U(G)). \end{aligned} \tag{5.81}$$

Hence, $\mathfrak{G}(U, \{\mu_t\})$ is a convex combination of elements of the closed convex cone $\mathcal{C}(U(G))$. By a similar argument $\mathfrak{W}(U, \{\mu_t\})$ belongs to $\mathcal{C}(U(G))$, as well.

The canonical form (5.76) of the superoperator $\mathfrak{G}(U, \{\mu_t\})$ follows from a direct calculation (hint: expand the self-adjoint operators $i\hat{X}_1, \dots, i\hat{X}_n$ with respect to an orthonormal basis in $\mathcal{B}_{\mathbb{R}}(\mathcal{H})$ including a multiple of the identity, and exploit the fact that $[a^{jk}]_{j,k=1}^n$ is a positive symmetric matrix). If $\{\mu_t\}_{t \in \mathbb{R}^+}$ is a Gaussian semigroup of measures, then the associated Lévy measure is identically zero and $[a^{jk}]_{j,k=1}^n \neq 0$. Therefore, in this case, $\mathfrak{W}(U, \{\mu_t\}) = 0$ and $\mathfrak{G}(U, \{\mu_t\})$ must be of the form (5.15).

Let us prove the last assertion of the theorem. First, if $\gamma_0(\mathfrak{U} - I) \neq 0$, choose a Lévy measure η (of the first kind) on G_* as a superposition of point mass measures in such a way that

$$\int_{G_*} (U \vee U(g) - I) d\eta(g) = \gamma_0(\mathfrak{U} - I), \quad (\eta(G_*) = \gamma_0); \quad (5.82)$$

otherwise set $\eta = 0$. Next, take vectors $\zeta_0, \zeta_1, \dots, \zeta_D$ in $\text{Lie}(G)$ such that

$$\zeta_0 \in \left(\pi_U^{-1}(-i\hat{P}_U^{-1}(\hat{H})) - \sum_{j=1}^n c^j(\eta)\xi_j \right), \quad \zeta_k \in \pi_U^{-1}(i\hat{P}_U^{-1}(\hat{F}_k)), \quad k = 1, \dots, D, \quad (5.83)$$

where \hat{P}_U is the orthogonal projection (with respect to the Hilbert–Schmidt scalar product) of $\mathcal{B}_{\mathbb{R}}(\mathcal{H})$ onto \mathcal{V}_U . Now, expand the vectors $\zeta_0, \zeta_1, \dots, \zeta_D$ with respect to the basis $\{\xi_1, \dots, \xi_n\}$ in $\text{Lie}(G)$: $\zeta_0 = \sum_{j=1}^n b^j \xi_j$, $\zeta_k = \sum_{l=1}^n d_{kl} \xi_l$, $k = 1, \dots, D$. At this point, one can check that

$$\begin{aligned} \mathfrak{L} = \mathfrak{G} + \gamma_0(\mathfrak{U} - I) &= \sum_{j=1}^n b^j [\hat{X}_j, (\cdot)] + \sum_{j,k=1}^n a^{jk} (\{\hat{X}_j \hat{X}_k, (\cdot)\} - 2\hat{X}_j(\cdot) \hat{X}_k) \\ &+ \int_{G_*} (U \vee U(g) - I) d\eta(g) + \sum_{j=1}^n c^j(\eta) [\hat{X}_j, (\cdot)], \end{aligned} \quad (5.84)$$

where $[a^{jk}]_{j,k=1}^n$ is the positive real matrix defined by

$$a^{jk} := \frac{1}{2} \sum_{l,m=1}^D \gamma_l \delta_{lm} d_{lj} d_{mk}. \quad (5.85)$$

Finally, let $\{\mu_t\}_{t \in \mathbb{R}^+}$ be the continuous convolution semigroup of measures associated with the representation kit $\{b^j, a^{jk}, \eta\}_{j,k=1}^n$. From formula (5.84) it follows that $\mathfrak{L} = \mathfrak{L}(U, \{\mu_t\})$.

The proof is complete. \square

Remark 5.3. Given any pair of representation kits $\{b^j, a^{jk}, \eta\}_{j,k=1}^n$ and $\{\tilde{b}^j, \tilde{a}^{jk}, \tilde{\eta}\}_{j,k=1}^n$ (of convolution semigroups of measures on G), for all $r, \tilde{r} \in \mathbb{R}^+$ one can define the set

$$r\{b^j, a^{jk}, \eta\}_{j,k=1}^n + \tilde{r}\{\tilde{b}^j, \tilde{a}^{jk}, \tilde{\eta}\}_{j,k=1}^n := \{rb^j + \tilde{r}\tilde{b}^j, ra^{jk} + \tilde{r}\tilde{a}^{jk}, r\eta + \tilde{r}\tilde{\eta}\}_{j,k=1}^n, \quad (5.86)$$

which is again the representation kit of a convolution semigroup of measures on G . Then, from theorem 5.1 it follows that the set

$$\begin{aligned} \mathcal{G}(U) &:= \{\mathfrak{L}(U, \{\mu_t\}) \\ &\in \mathcal{L}(\mathcal{H}) : \{\mu_t\}_{t \in \mathbb{R}^+} \text{ continuous convolution semigroup of measures on } G\} \end{aligned} \quad (5.87)$$

of all generators of twirling semigroups associated with the representation U is a convex cone contained in $\mathcal{C}(\mathcal{V})$. Note that the convex cone $\mathcal{G}(U)$ is not ‘pointed’ (i.e. it is a ‘wedge’), unless the representation U is trivial. In fact, we have that

$$\mathcal{G}_0(U) := \mathcal{G}(U) \cap (-\mathcal{G}(U)) = \{i[\hat{H}, (\cdot)] : \hat{H} \in \mathcal{V}_U\}. \quad (5.88)$$

The set $\mathcal{G}_0(U)$ is the ‘lineality space’ [44] of the convex cone $\mathcal{G}(U)$. It is a vector space contained in the closed cone $\text{cone}_0(\overline{\mathcal{V}} \vee \overline{\mathcal{V}} - I)$. The lineality space $\mathcal{G}_0(U)$ is the smallest face (extreme subset) of the convex cone $\mathcal{G}(U)$, namely it is a face of $\mathcal{G}(U)$, and any other face of $\mathcal{G}(U)$ contains $\mathcal{G}_0(U)$. Moreover, the following decomposition holds:

$$\mathcal{G}(U) = \mathcal{G}_0(U) + \mathcal{G}_1(U), \tag{5.89}$$

where $\mathcal{G}_1(U)$ is the pointed cone defined by $\mathcal{G}_1(U) := \{0\} \cup (\mathcal{G}(U) \setminus \mathcal{G}_0(U))$.

Recalling the second assertion of proposition 5.1, and applying the last assertion of theorem 5.1 to the defining representation of the group $\text{SU}(\mathbb{N})$, we get the following result.

Corollary 5.1. *Let \mathcal{H} be a finite-dimensional Hilbert space. Then, every twirling semigroup acting in $\mathcal{B}(\mathcal{H})$ is a random unitary semigroup and, conversely, every random unitary semigroup acting in $\mathcal{B}(\mathcal{H})$ arises as a twirling semigroup.*

6. Conclusions, final remarks and perspectives

In this paper, we have studied the main properties of a well-characterized class of semigroups of (super) operators acting in Banach spaces of trace-class operators. These semigroups of superoperators—that we have called *twirling semigroups*—are associated in a natural way with the pairs of the type $(U, \{\mu_t\}_{t \in \mathbb{R}^+})$, where U is a projective representation of a l.c.s.c. group G and $\{\mu_t\}_{t \in \mathbb{R}^+}$ is a continuous convolution semigroup of (probability) measures on G . In section 4, we have proved that the twirling semigroups are *quantum dynamical semigroups*. Hence, they describe the dynamics of a class of *open quantum systems*. It is interesting to note that every twirling semigroup can be regarded as the restriction to $\mathcal{B}_1(\mathcal{H})$ of a (continuous) contraction semigroup in the Hilbert space $\mathcal{B}_2(\mathcal{H})$, see remark 4.4.

Next, in order to provide a complete characterization of the twirling semigroups, we have studied their infinitesimal generators. In this paper, as a first step, we have analyzed in detail the case where G is a Lie group and U is a finite-dimensional, smooth (equivalently, continuous), unitary representation. However, we stress that, thanks to Nelson’s theory of analytic vectors [45], one can extend some of the results of section 5 to the case where U is a generic strongly continuous unitary representation of a Lie group by taking care of the domains of the (in general, unbounded) infinitesimal generators of the associated twirling semigroups. This task will be accomplished elsewhere [46].

The main technical tool that we have exploited for proving the main result of section 5—i.e. theorem 5.1—is the classical Lévy–Kintchine formula, which, in the mathematical literature [5], is shown to hold for suitable classes of functions ‘vanishing at infinity’—in particular, for smooth functions with compact support—and is not *directly* applicable to our context (that involves, in general, *bounded* smooth functions on a Lie group G). Indeed, as the reader will have noticed, it has been necessary to prove lemma 5.3 in order to use the Lévy–Kintchine formula ‘as if the smooth function $G \ni g \mapsto \langle \phi, U(g)\hat{A}U(g)^*\psi \rangle$, $\hat{A} \in \mathcal{B}(\mathcal{H}) (= \mathcal{B}_1(\mathcal{H}))$, \mathcal{H} being finite dimensional), $\phi, \psi \in \mathcal{H}$, belonged to $C_c^\infty(G; \mathbb{C})$ ’, which, in general, is not the case. Moreover, as the reader may verify, to derive the expression of the infinitesimal generator of the twirling semigroup associated with the pair $(U, \{\mu_t\}_{t \in \mathbb{R}^+})$ is simpler if one assumes that $\{\mu_t\}_{t \in \mathbb{R}^+}$ is a *Gaussian* semigroup of measures (to this aim, one can exploit the defining condition (5.18)), i.e. if $\{\mu_t\}_{t \in \mathbb{R}^+}$ is the distribution associated with a Brownian motion on G .

In addition to these technical remarks, it is also worth observing that twirling semigroups are a natural source of *covariant* quantum dynamical semigroups. In fact, let $\{\mathfrak{S}_t\}_{t \in \mathbb{R}^+}$ be the

twirling semigroup associated with the pair $(U, \{\mu_t\}_{t \in \mathbb{R}^+})$. Consider the set

$$G_0(U, \{\mu_t\}) := \{g \in G : \mathfrak{S}_t(U \vee U(g)\hat{A}) = U \vee U(g)(\mathfrak{S}_t\hat{A}), \forall t \in \mathbb{R}^+, \forall \hat{A} \in \mathcal{B}_1(\mathcal{H})\}. \tag{6.1}$$

As the reader may easily check, $G_0(U, \{\mu_t\})$ is a closed subgroup of G . This subgroup includes the set

$$G_0(U) := \{g \in G : U \vee U(gh) = U \vee U(hg), \forall h \in G\}, \tag{6.2}$$

which is a closed normal subgroup of G containing the center of G . For instance, in the case where U is a projective representation of an Abelian group G , we have

$$G = G_0(U, \{\mu_t\}) = G_0(U). \tag{6.3}$$

Now, let \check{G} be any subgroup of $G_0(U, \{\mu_t\})$, and let $\check{U} : \check{G} \rightarrow \mathcal{U}(\mathcal{H})$ be the projective representation defined by

$$\check{U}(g) = U(g), \quad \forall g \in \check{G}. \tag{6.4}$$

Then, we have that

$$\mathfrak{S}_t(\check{U}(g)\hat{A}\check{U}(g)^*) = \check{U}(g)(\mathfrak{S}_t\hat{A})\check{U}(g)^*, \quad \forall t \in \mathbb{R}^+, \quad \forall g \in \check{G}, \quad \forall \hat{A} \in \mathcal{B}_1(\mathcal{H}), \tag{6.5}$$

namely—by definition, see [14]—the quantum dynamical semigroup $\{\mathfrak{S}_t\}_{t \in \mathbb{R}^+}$ is covariant with respect to the representation \check{U} .

Another issue that is worth discussing is the characterization of the twirling superoperators that are *Markovian channels* [47, 48] (we would prefer the term *embeddable channels*), i.e. that are members of quantum dynamical semigroups. Precisely, a twirling superoperator \mathfrak{S} is a Markovian channel if $\mathfrak{S} = \mathfrak{S}_1$, for some quantum dynamical semigroup $\{\mathfrak{S}_t\}_{t \in \mathbb{R}^+}$ (not necessarily a twirling semigroup). Clearly, if the twirling superoperator \mathfrak{S} is induced by a pair (U, μ) (which is, in general, not unique) such that the probability measure μ is *embeddable*—namely $\mu = \mu_1$ for some continuous convolution semigroup of measures $\{\mu_t\}_{t \in \mathbb{R}^+}$ (see [5])—then it is a Markovian channel and a member of the twirling semigroup associated with the pair $(U, \{\mu_t\}_{t \in \mathbb{R}^+})$. However, whether *every* twirling superoperator which is a Markovian channel is a member of a twirling semigroup seems to be an interesting open problem. The investigation of this problem, in the light of known results about the relation between embeddable and *divisible* probability measures [5], may also lead to a deeper understanding of the relation between Markovian and *divisible* channels [48].

Finally, we note that, if the representation $U : G \rightarrow \mathcal{U}(\mathcal{H})$ is genuinely projective, by considering a central extension [30] G_{ext} of the circle group \mathbb{T} by G one can always represent any twirling semigroup associated with U as a twirling semigroup associated with a standard unitary representation of G_{ext} (consider that every convolution semigroup of measures on G can be trivially extended to G_{ext}).

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